Analytic Geometry and Calculus I – Exam 1  
Summer 2006

Show all work for full credit. You may use a basic function calculator, but not a graphing or scientific calculator. No notes or books are allowed.

<table>
<thead>
<tr>
<th>page</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>points</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(10 pts) 1. Sketch the graph of the following function and use it to determine the values of \( a \) for which \( \lim_{x \to a} f(x) \) exists.

\[
f(x) = \begin{cases} 
  x^2 + 1, & \text{if } x < -1 \\
  2x, & \text{if } -1 \leq x < 2 \\
  -x + 4, & \text{if } x \geq 2 
\end{cases}
\]

![Graph of the function](image)

\[
\lim_{x \to a} f(x) \text{ exists when } x \neq -1, \text{ and } x \neq 2.
\]

At \( x = 1 \) & \( x = 2 \), the left-hand limits don't equal the right-hand limits.

Ans: \((-\infty, -1) \cup (-1, 2) \cup (2, \infty)\)
(30 pts) 2. Evaluate the following limits (if the limit does not exist, explain why).

(a) \[ \lim_{x \to 1} \left( x^2 + \frac{3}{x+2} \right) = \lim_{x \to 1} \left( x^2 + \frac{3}{1+2} \right) = 1 + 1 = \boxed{2} \]

(b) \[ \lim_{x \to 3} \frac{2x^2 - 5x - 3}{x - 3} = \lim_{x \to 3} \frac{(x-3)(2x+1)}{x-3} = \lim_{x \to 3} 2x+1 = 2 \cdot 3 + 1 = \boxed{7} \]

(c) \[ \lim_{h \to 0} \frac{(3+h)^2 - 9}{h} = \lim_{h \to 0} \frac{9+6h+h^2 - 9}{h} = \lim_{h \to 0} \frac{6h+h^2}{h} = \lim_{h \to 0} (6+h) = 6+0 = \boxed{6} \]

(d) \[ \lim_{x \to 6} \frac{x^2 - 36}{\sqrt{x} - \sqrt{6}} = \lim_{x \to 6} \frac{(x+6)(x-6)}{\sqrt{x} - \sqrt{6}} = \lim_{x \to 6} \frac{(x+6)(\sqrt{x}+\sqrt{6})(\sqrt{x}-\sqrt{6})}{\sqrt{x}-\sqrt{6}} = \lim_{x \to 6} (x+6)(\sqrt{x}+\sqrt{6}) = (6+6)(\sqrt{6}+\sqrt{6}) = 12 \cdot 2\sqrt{6} = 24\sqrt{6} \]

(e) \[ \lim_{x \to 3} \frac{|x-3|}{x-3} = \lim_{x \to 3^+} \frac{x-3}{x-3} = \lim_{x \to 3^+} |x-3| = 1 \]

\[ \lim_{x \to 3^-} \frac{|x-3|}{x-3} = \lim_{x \to 3^-} -\frac{x-3}{x-3} = \lim_{x \to 3^-} |x-3| = -1 \]

Thus, since \(1 \neq -1\), \[ \lim_{x \to 3} \frac{|x-3|}{x-3} \text{ DNE.} \]

(f) \[ \lim_{y \to \infty} \frac{2-3y^2}{5y^2 + 4y} = \lim_{y \to \infty} \frac{\frac{2}{y^2} - 3}{\frac{5}{y^2} + \frac{4}{y}} = \lim_{y \to \infty} \frac{\frac{2}{y^2}}{\frac{5}{y^2} + \frac{4}{y}} = \lim_{y \to \infty} \frac{2}{5} = \frac{0 - 3}{5 + 0} = \boxed{-\frac{3}{5}} \]
(10 pts) 3. By calculating an appropriate limit, find the slope of the tangent line to the graph of the function \( f(x) = x^2 \) at \((3, 9)\).

\[ m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{f(3+h) - 9}{h} = \lim_{h \to 0} \frac{(3+h)^2 - 9}{h} = 6. \]

From 2.c.

Alternate Method: \( \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} = \lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} x + 3 = 6. \)

(10 pts) 4. Explain why the function is discontinuous at 1 and sketch the graph.

\[ f(x) = \begin{cases} 
  x^2, & \text{if } x \leq 1 \\
  2x, & \text{if } x > 1
\end{cases} \]

\[ \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2x = 2(1) = 2 \]

\[ \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 = (1)^2 = 1 \]

\[ \lim_{x \to 1^+} f(x) \neq \lim_{x \to 1^-} f(x), \]

so \( \lim_{x \to 1} f(x) \) doesn't exist,

so \( f \) is discontinuous at 1.
(10 pts) 5.

(a) From the graph of \( f \), state the numbers at which \( f \) is discontinuous and explain why.

\( f \) is discontinuous at \( x = -3 \) : \( \lim_{{x \to -3^-}} f(x) \neq \lim_{{x \to -3^+}} f(x) \).

\( f \) is also discontinuous at \( x = 2 \) : \( \lim_{{x \to 2^-}} f(x) \neq f(2) \) and at \( x = 4 \) : \( \lim_{{x \to 4^-}} f(x) \neq \lim_{{x \to 4^+}} f(x) \).

\( f \) is also discontinuous at \( x = -6 \) & \( x = 6 \) because only one one-sided limit exists. (\( f \) is left continuous at \( x = 6 \) because \( \lim_{{x \to 6^-}} f(x) = f(6) \).)

(b) From the graph of \( f \), state the open intervals on which \( f \) is continuous.

\( \text{Ans: } (-6,-3) \cup (-3,2) \cup (2,4) \cup (4,6) \)

Note: If we did not have the word open, we have to think about right & left continuity. Some would say \( f \) is continuous on \( (-6,-3) \cup (-3,2) \cup (2,4) \cup (4,6) \) and others: \( (-6,-3) \cup [-3,2) \cup (2,4] \cup [4,6] \).
(10 pts) 6. The graph shows the position function of a car. Use the shape of the graph to explain your answers to the following questions.

(a) Was the car going faster at $A$ or $B$?

(b) At which point(s) was the car slowing down? $A$ & $B$

(c) At which point(s) was the car speeding up? $D$

(d) What happened around point $C$? The car is stopped — it's not moving.
(10 pts) 7. Find the constant $c$ that makes $g$ continuous on $(-\infty, \infty)$.

$$g(x) = \begin{cases} 
      cx + 3c, & \text{if } x < 2 \\
      5x, & \text{if } x \geq 2
\end{cases}$$

We need

$$\lim_{x \to 2^+} g(x) = \lim_{x \to 2^-} g(x).$$

$$\lim_{x \to 2^+} 5x = \lim_{x \to 2^-} cx + 3c$$

$$10 = 2c + 3c$$

$$10 = 5c$$

$$\overline{2 = c}$$

$$\lim_{x \to 2} g(x) = \lim_{x \to 2} 5x = 10$$

so

$$g(2) = 5 \cdot 2 = 10,$$ 

$$\lim_{x \to 2} g(x).$$

(10 pts) 8. Let $f(x) = x^2 + 5$. Explain why there is a number $c$ such that $f(c) = 12$.

$$f(2) = 2^2 + 5 = 9$$

$$f(3) = 3^2 + 5 = 14$$

$$9 < 12 < 14$$

$$f(2) < 12 < f(3).$$

Since $f(x)$ is continuous, by the Intermediate Value Theorem (IVT), there exist a $c$ in $(2, 3)$ such that $f(c) = 12$. 