Practice Midterm # 2 (The actual midterm will be shorter!)

Math 220, Fall 2014

You will not be allowed to use any type of calculator whatsoever, you will not be allowed to have any other notes, the test will be closed book, and there is no escape. The actual test will be graded in red ink! There will be no mercy for the weak. Mathematics is cumulative. Deal with it. What you don’t know will hurt you. You need to be able to make simple and/or standard simplifications. (Especially where I emphasize!) In order to get credit or partial credit, your work must make sense.

I strongly suggest that you take this practice test under the conditions of the actual test! (Except that you might not do it all at once since it is longer than the test will be.)

1. Find the first and second derivatives of the following functions. Simplify where appropriate.

(a) 
\[ y = x^2 e^{5x} . \]

(b) 
\[ y = x^3 \cos(x^2 + 4) . \]

(c) 
\[ y = \frac{\sqrt{x^2 + 1}}{x} \quad \text{Simplify!} \]

2. Differentiate the following functions. Simplify where appropriate.

(a) 
\[ y = \arccot(3 \arccos(4x)) . \]

(b) 
\[ y = \ln(1 + xe^x) . \]

(c) 
\[ y = (\csc(2x))(\cot(3x))(\sec(4x)) . \]

(d) 
\[ y = \frac{x^2 + 4}{\sqrt{x^2 + 1}} \quad \text{Simplify!} \]
(e) \[ y = \frac{\sqrt{x^2 + 3}}{x^2 + 2} \] Simplify!

(f) \[ y = \frac{e^{2x} \sqrt{x^2 + 1}}{x} \] Simplify!

(g) \[ y = \frac{x^3 \cos(3x)}{1 + x^2 e^{5x}} \cdot 

(h) \[ y = \sin(5 \sin(6 \cos(7 x))) \]

(i) \[ y = \sin(x^2 \sin(x^3 \cos(4x))) \]

(j) \[ y = e^{\left(\frac{1 + x \sin x}{2 + x \cos x}\right)} \]

3. Derive the formulas for the derivatives of \( \tan x \), \( \cot x \), \( \sec x \), \( \csc x \), \( \arcsin x \), and \( \arccos x \).

4. Find \( \frac{dy}{dx} \) for the following implicitly defined functions at the point in question.

(a) \[ x^2 + y^2 = 25 \text{ at the points } (4, 3) \text{ and } (4, -3). \]

(b) \[ x^2 + y^2 = 169 \text{ at the points } (5, 12) \text{ and } (5, -12). \]
   (You can check your work for the last two problems by solving for \( y \) and differentiating in the “old-fashioned” way. Be careful about the sign of the square root.)

(c) \[ x^3 y^2 - x^2 y^3 + xy = 6 \text{ at the point } (2, 1). \]

(d) \[ ye^{xy} = 2e^2 \text{ at the point } (1, 2). \]
(e) \[
y \sin \left( \frac{\pi x}{4} + \frac{\pi y}{6} \right) = 2 \quad \text{at the point} \quad (-2, 4).
\]

5. Find \( \frac{dy}{dx} \) using logarithmic differentiation. (i.e. Take the logarithm and then differentiate, and then solve for \( \frac{dy}{dx} \).)

(a) \[
y = x^x.
\]

(b) \[
y = (2x)^{\sin x}.
\]

(c) \[
y = (3x + 4)^{5x+6}.
\]

(d) \[
y = \frac{\sin^7(5x)(\ln(1 + x^6))^{1/3}}{(x^2 + 5)^9 e^{(x^4+1)}}.
\]

(You don’t need to simplify this one all the way.)

6. For this problem you should know that the volume of a ball of radius \( r \) is \( \frac{4}{3} \pi r^3 \) and its surface area is \( 4 \pi r^2 \). You will also need:

\[
\begin{align*}
2.1^2 &= 4.41, & 2.5^2 &= 6.25, & 2.1^3 &= 9.261, & 2.5^3 &= 15.625.
\end{align*}
\]

(Leave \( \pi \) as \( \pi \) everywhere it occurs!) Give the units of your final answers!

(a) Find the average rate of change of the area of a square with respect to its side length, \( s \), as \( s \) changes from

i. 2 in to 3 in
ii. 2 in to 2.5 in
iii. 2 in to 2.1 in

(b) Find the instantaneous rate of change of the area of the square with respect to its side length when \( s = 2 \) in. (Note that this is half of the perimeter of the square.)

(c) Find the average rate of change of the volume of a ball with respect to its radius, \( r \), as \( r \) changes from
i. 2 in to 3 in
ii. 2 in to 2.5 in
iii. 2 in to 2.1 in

(d) Find the instantaneous rate of change of volume of the ball with respect to radius when \( r = 2 \) in. (Note that this is the surface area of the sphere.)

(e) Find the average rate of change of the volume of a cube with respect to its side length, \( s \), as \( s \) changes from
   i. 2 in to 3 in
   ii. 2 in to 2.5 in
   iii. 2 in to 2.1 in

(f) Find the instantaneous rate of change of volume of the cube with respect to its side length when \( s = 2 \) in. (Note that this is half of the surface area of the cube.)

7. A stone is dropped onto a lake, creating a circular ripple that travels outward at a speed of 50 in/sec. Find the rate at which the area within the circular ripple is increasing after 1 sec, 3 sec, and 5 sec. Give the units of your final answers!

8. A tank which holds 1000 gallons of water drains according to the formula:

\[ V = 1000 \left(1 - \frac{t}{20}\right)^2 \quad 0 \leq t \leq 20 \]

where \( V \) is volume of water remaining in the tank at time \( t \) (given in minutes). What is the rate at which water is flowing out of the tank after 5 min, 10 min, 15 min, and 20 min? Give the units of your final answers!

9. A machine inflates a spherical balloon at a rate of 5in\(^3\)/sec. How fast is the radius increasing at the moment that the radius is 4in? How fast is the surface area increasing at the same moment? Give the units of your final answers!

10. A 15ft ladder is leaning against a wall. The bottom of the ladder slides away at a constant rate of 2ft/sec. Find the rate at which the top of the ladder slides down the wall at the instant when the bottom of the ladder
is 9ft from the wall. Give the units of your final answers! According to your model, what happens to the speed of the top of the ladder as the bottom of the ladder gets close to being (but less than) 15ft away from the wall? What can you conclude?

11. A spotlight lying on the ground shines on a wall 25ft away. A 10ft tall giraffe walks from the wall toward the spotlight at a speed of 3ft/sec. At what rate is the height of its shadow on the wall increasing at the instant when it is 15ft away from the light? Give the units of your final answers!

12. A hose fills a conical tank (a right circular cone with the “tip” of the cone at the bottom) with the radioactive liquid “New Jerseyium” at a rate of 10ft$^3$/min. The tank has height $h = 50$ft and radius $r = 40$ft at the top. How fast is the fluid rising at the moment that the height is 5ft? Give the units of your final answers! (Note that the volume, $V$, of a right circular cone with height, $h$, and radius, $r$, is given by the formula:

$$V = \frac{1}{3}\pi r^2 h.$$ Note also that $h$ and $r$ are related by similar triangles in a problem of this type.)

13. A plane leaves Phoenix at 3 PM travelling North at 400miles/hour. At 4 PM another plane leaves Phoenix travelling East at 300miles/hour. At what rate is the distance between these planes increasing at 5 PM? What if the first plane started at 2 PM instead of 3 PM? Give the units of your final answers!

14. Particle on a curve problems...

(a) If $y = x^2 + 4x + 2$, and $\frac{dx}{dt} = 5$, then what is $\frac{dy}{dt}$ when $x = 1$?

(b) If $y = 2\sqrt{x}$, and $\frac{dx}{dt} = 4$, then what is $\frac{dy}{dt}$ when $x = 9$?

(c) If $y = e^{4x}$, and $\frac{dx}{dt} = 3$, then what is $\frac{dy}{dt}$ when $x = 2$?

(d) If $y = \sqrt{x^2 + 1}$, and $\frac{dx}{dt} = 17$, then to what does $\frac{dy}{dt}$ converge as $x \to \infty$?

15. On planet X if a stone is thrown upward with an initial velocity of $v_0$ (given in feet per second), then the height of the stone at time $t$ is given
by \( h(t) = v_0 t - 2t^2 \). (I am assuming that the stone is being thrown from height zero.)

(a) If \( v_0 = 12 \) feet per second, then how high will the stone get?
(b) What will the velocity be when the stone first passes through the height of 16 feet?
(c) What will the velocity be the second time that the stone has height 16?

16. (a) Use linear approximation (from \( x = 25 \)) to estimate:
   i. \( \sqrt{25.1} \)
   ii. \( \sqrt{25.01} \)
   iii. \( \sqrt{24.9} \)
   iv. \( \sqrt{24.99} \)

(b) Observe that the second derivative of the square root function is negative, and use this to predict whether the actual values above are more or less than your estimates. (Hint: where is the function in relation to the tangent line? Think about the graph!)

17. (a) Use linear approximation (from \( x = 1 \)) to estimate:
   i. \( 1.1^{-1} \)
   ii. \( 1.01^{-1} \)
   iii. \( 0.9^{-1} \)
   iv. \( 0.99^{-1} \)

(b) Observe that the second derivative of the function \( x^{-1} \) is positive, and use this to predict whether the actual values above are more or less than your estimates. (Same Hint...)

18. Use linear approximation to estimate the amount of paint needed to apply a coat of paint .04 cm thick to a hemispherical dome with radius 50 cm.

19. The radius of a ball is measured to be 20 cm with a possible error of .2 cm.

   (a) Use linear approximation to estimate the maximum error in the calculated surface area.
(b) Use linear approximation to estimate the maximum error in the calculated volume.