Recall from §2.2: if $f$ is a function and there is a number $L$ such that

$$\lim_{{x \to \infty}} f(x) = L \quad \text{or} \quad \lim_{{x \to -\infty}} f(x) = L$$

then the graph of $f$ is snuggling up to the horizontal line $y = L$ called a horizontal asymptote. Definition 3 and Figures 14, 15, p. 144 make this precise.

Note: The $L$ here is a number, not $-\infty$ or $\infty$!

Typically $\lim_{{x \to -\infty}} f(x)$ looks like

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Figure 1}
\end{figure}

For example, $L = 1$ and

$$f(x) = 1 - \frac{1}{x} \quad \text{or} \quad f(x) = 1 + \frac{1}{x}$$

(approach from below) (approach from above).

But, the graph of $f$ can cross this line, even infinitely often. For example,
\[ f(x) = \frac{1}{x}\sin x \]

As \( x \to 0 \), \( \frac{1}{x} \to 0 \), and the factor \( \sin x \) causes the sign to change every time \( x \) moves \( \pi \) further right, since \( \sin (x + \pi) = -\sin x \).

The whole preceding discussion applies "on the left," that is, to (1)_ as well as to (1)_.

**Ex. 1** No (non-constant) polynomial \( P(x) \) has any horizontal asymptotes. For if the leading term of \( P(x) \) is \( cx^k \) where \( k > 1 \) and coefficient \( c \) is not 0, then this term overwhelms all others when \( |x| \) is large and

\[
\lim_{x \to \infty} P(x) = \lim_{x \to \infty} cx^k
\]

and this limit is always infinite. More precisely,

\[
(2)_+ \quad \lim_{x \to \infty} cx^k = \begin{cases} \infty & \text{if } c > 0 \\ -\infty & \text{if } c < 0 . \end{cases}
\]

Going left things are a little more involved (why?):

\[
(2)_- \quad \lim_{x \to -\infty} cx^k = \begin{cases} \infty & \text{if } c > 0, \ k \ \text{even} \\ -\infty & \text{if } c > 0, \ k \ \text{odd} \\ -\infty & \text{if } c < 0, \ k \ \text{even} \\ \infty & \text{if } c < 0, \ k \ \text{odd} . \end{cases}
\]
Consequently, every polynomial graph has one of four forms:

The dotted portions can involve numerous peaks and valleys, but ultimately (i.e., when \(|x|\) is large) two "arms" will sprout, in one of these four configurations.

In §2.2 Lecture p. 10 we found the criteria for when a rational function \(R(x) = \frac{N(x)}{D(x)}\) has a horizontal asymptote. To review it briefly, let us write
Numerator polynomial and Denominator polynomial

\[ N(x) = c_n x^n + \text{lower powers of } x \]

\[ D(x) = d_m x^m + \text{lower powers of } x. \]

As noted above, when \( |x| \) is large the leading term overwhelms the others, so

\[ N(x) \approx c_n x^n \quad \text{(approximately equal)}, \quad D(x) \approx d_m x^m, \]

(3) \[ R(x) \approx \frac{c_n x^n}{d_m x^m} = \frac{c_n}{d_m} x^{n-m}, \text{ when } |x| \text{ large}. \]

Consequently, as \( |x| \to \infty \) the behavior of \( R(x) \) is determined by whether the number \( \frac{c_n}{d_m} \) (let’s call it \( a \) for short) is positive or negative and whether exponent \( n-m \) is positive, \( 0 \), or negative. Precisely,

(4) \[ \lim_{x \to \infty} R(x) = \begin{cases} 
\infty & \text{if } c>0, n>m \\
-\infty & \text{if } c<0, n>m \\
a \neq 0 & \text{if } n=m \\
0 & \text{if } n<m 
\end{cases} \quad \deg N > \deg D \]

In terms of \( a, n \) and \( m \) what are the possibilities for

(4) \[ \lim_{x \to -\infty} R(x) ? \]

(Now whether \( n-m \) is even or odd affects the sign of the limit: \( \pm \infty, \pm a \).)
2.6 “Dividing by \( x \) as we did above is useful in many contexts, because \( |x| \to \infty \) means \( t = \frac{1}{x} \to 0 \) and 0 is easier to deal with than \( \infty \). E.g.,

\[
\frac{\sqrt{x^2 + 1}}{x+2} = \frac{\frac{1}{x}\sqrt{x^2 + 1}}{\frac{1}{x}(x+2)} = \frac{\sqrt{\frac{1}{x^2} \cdot (x^2 + 1)}}{1 + \frac{2}{x}} = \frac{\sqrt{1 + \frac{1}{x^2}}}{1 + \frac{2}{x}}
\]

Consequently

\[
\lim_{x \to \infty} \frac{\sqrt{x^2 + 1}}{x+2} = \lim_{x \to \infty} \frac{\sqrt{1 + \frac{1}{x^2}}}{1 + \frac{2}{x}} = \lim_{t \to 0} \frac{\sqrt{1 + t^2}}{1 + 2t} = 1, \text{ by the quotient limit law (5. on p. 104).}
\]

A function can have two different horizontal asymptotes, but at most two (why?)

Ex. 3 \( g(x) = \tan^{-1} x \)

\( = \arctan x \)

whose graph is shown satisfies

\[
(5) \lim_{x \to -\infty} g(x) = -\frac{\pi}{2}, \lim_{x \to \infty} g(x) = \frac{\pi}{2}.
\]

Recall that for a one-to-one function \( f \) we get the graph of \( f^{-1} \) by rotating the graph paper \( 180^\circ \) over the diagonal line \( y = x \). In the process
vertical lines become horizontal lines, so if \( x = a \) is the equation of a vertical asymptote for \( f \), then \( y = a \) is the equation of a horizontal asymptote for \( f^{-1} \). Ex. 3 illustrates this with \( a = \frac{\pi}{2} \), \( f(x) = \tan x \).

In fact, it’s from knowing

\[
\lim_{x \to \frac{\pi}{2}^-} \tan x = \infty
\]

that we get (5). Do you see why (6) is true, i.e., why Figure 5 is correct? Here’s how: You give me a very large number \( N (>1) \), and I need to show you a number \( \delta > 0 \) so small that \( \tan x > N \) whenever \( 0 < \frac{\pi}{2} - x < \delta \).

Since \( \cos x \) is decreasing on the interval \([0, \pi]\) (see graph above), if

\[
\frac{\pi}{2} - \frac{1}{2N} < x < \frac{\pi}{2}
\]

then

\[
0 < \cos x < \cos \left( \frac{\pi}{2} - \frac{1}{2N} \right) = \sin \left( \frac{1}{2N} \right) < \frac{1}{2N}.
\]

Remember (Lecture §2.3 p.5 or p.212 of text) that \( \sin \theta < \theta \) for all \( \theta > 0 \). Re-writing ineq. (8)
\[ \frac{1}{\cos x} > 2N. \]

Now \( N > 1 \) entails
\[ \frac{\pi}{2} - \frac{1}{2N} > \frac{\pi}{3} \quad \text{(check!)} , \]
so the increasing graph of sine shows that
\[ \frac{1}{2} = \sin \frac{\pi}{3} < \sin \left( \frac{\pi}{2} - \frac{1}{2N} \right) < \sin x . \]

Putting the pieces together,
\[ \tan x = \frac{\sin x}{\cos x} > \frac{1}{2} \cdot \frac{1}{\cos x} > \frac{1}{2} \cdot 2N = N \]
whenever \( x \) satisfies (7). This confirms (6).

Ex. 4. No rational function can have two different horizontal asymptotes.

For, according to (4)\(_+\) and (4)\(_-\), there is a horizontal asymptote only if \( \deg N < \deg D \), then the sole horizontal asymptote is \( y = 0 \)

or

if \( \deg N = \deg D \), then the sole horizontal asymptote is \( y = \frac{c_N}{d_m} = \text{ratio of leading terms} \).
In this section we show the equivalence of two important issues:

1. How to get a tangent line to a graph from its secant lines?

2. How to get an instantaneous rate from average rates?

Limits are at the heart of both questions. Suppose the function \( d(t) \) describes my displacement (in feet), positive or negative, along a line from some starting point after elapsed time \( t \) (seconds). Suppose I want to know my speed at the point \( x_0 = d(t_0) \) on my path. At time \( t > t_0 \), I'll be at the new position \( d(t) \) and will have covered the (positive or negative) distance \( d(t) - d(t_0) \) in time \( t - t_0 \). So my average speed (= distance/time) over time interval \([t_0, t]\) is

\[
\frac{d(t) - d(t_0)}{t - t_0} = \text{average speed over } [t_0, t].
\]
As \( t \to t_0^+ \) these averages over ever shorter time spreads should converge to a number that deserves to be regarded as my speed (or velocity) at the instant \( t_0 \). Denote it \( v(t_0) \): 

\[
v(t_0) = \lim_{{t \to t_0^+}} \frac{d(t) - d(t_0)}{t - t_0}.
\]

Needless to say, we don't get this limit by merely inserting \( t = t_0 \) into the quotient! Had I looked back to slightly earlier times \( t < t_0 \), computed average speeds, and let \( t \to t_0^- \), the same limit would have been found, unless at \( t_0 \) sudden acceleration had occurred. If both one-sided limits exist and they're equal we write

\[
(2)' \quad v(t_0) = \lim_{{t \to t_0}} \frac{d(t) - d(t_0)}{t - t_0}
\]

and call this limit my instantaneous velocity, at the instant \( t_0 \). This answers question (2), theoretically. (But hey, beyond mere counting all mathematics is theoretical—it occurs only in the mind of man, not in nature.)
Now we'll solve the apparently quite different problem (1), by essentially the same reasoning!

Consider a point \((x_0, y_0) = (x_0, f(x_0))\) on the graph of a function \(f\). (Figure 1)

Any nearby point \((x, y) = (x, f(x))\) determines a secant (or chord) line through \((x_0, y_0)\). Clearly

\[
(3) \quad \text{secant slope} = \frac{y - y_0}{x - x_0} = \frac{f(x) - f(x_0)}{x - x_0}.
\]

[This formula is the same whether \(x < x_0\) or \(x > x_0\) but Figure 1 shows \(x > x_0\). For obvious reasons, such a ratio is called a difference quotient, for \(f\).]

If the curve has a tangent at \((x_0, y_0)\), then as \(x \to x_0\)
the secant line appears to move into the tangent line through \((x_0, y_0)\), whose slope is consequently the limit of the secant's slope, that is,

\[
\text{slope of tangent at } (x_0, f(x_0)) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0}
\]

if there is a tangent at all! In that case,
by modifying Figure 1 to have $x < x_0$, it appears that the limit of (3) exists as $x \to x_0^-$ as well, and again gives the slope of the tangent. In summary

(4) \[
\begin{align*}
\text{If the graph of } f(x) \text{ has a tangent at } (x_0, f(x_0)) \\
\text{then slope of tangent } &= \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0}.
\end{align*}
\]

If it is finite, this limit is called

(5) \[
\begin{align*}
\text{the derivative at } x_0 \text{ of the function } f(x), \text{ and} \\
\text{the symbol for it is } f'(x_0) \text{ [read "f prime of } x_0\text{"].
}\end{align*}
\]

Note all the similarities with the velocity problem: (3) is the average rate of change of $f(x)$ over an interval of $x$-values and (4) is its "instantaneous" rate at $x_0$. If $x$ is called $t$ and $f$ is called $d$, this is the velocity problem, and Figure 1 is the graph of displacement vs. time.

In the language of (5)

(6) \[
\begin{align*}
\text{Velocity } v(t_0) \text{ at instant } t_0 \text{ is the derivative} \\
\text{of the displacement function } d(t) : \\
v(t_0) &= d'(t_0).
\end{align*}
\]

Tangents (hence derivatives) do not always exist!
Ex. 1 Consider the absolute-value function
\[ f(x) = |x| \]
at point \( x_0 = 0 \). All secants on the right have slope
\[ \frac{f(x) - f(x_0)}{x - x_0} = \frac{|x|}{x} = \frac{x}{x} = 1 \quad \text{if } x > x_0 = 0 \]
while all secants on the left have slope
\[ \frac{f(x) - f(x_0)}{x - x_0} = \frac{|x|}{x} = \frac{-x}{x} = -1 \quad \text{if } x < x_0 = 0. \]
So, \( \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \) does not exist. [The quotient has left-hand and right-hand limits, but they are different.] And the graph shows clearly a corner at \((0, 0)\). In a sense there is no tangent line at \((0, 0)\), but in the sense of "line touching the graph only at point \((0, 0)\)" there are infinitely many such lines!

Ex. 2 If we adopt the definition of tangent as "touching at only one point", the graph of a function can have a vertical tangent line at some point. Since vertical lines experience positive "rise" over 0 "run", we think of them having infinite slope (= rise-to-run ratio). Consequently
if the tangent is vertical, we expect the secant slopes \((3)\) to have limit \(\infty\). A good example is
\[
f(x) = \sqrt[3]{x} = x^{1/3}
\]
at point \(x_0 = 0\).
An accurate graph suggests the y-axis is tangent at the point \((0, 0)\). To confirm this, note that the slopes of the secants are
\[
\frac{y - y_0}{x - x_0} = \frac{y}{x} = \frac{x^{1/3}}{x} = \frac{1}{x^{2/3}} = \left(\frac{1}{x^2}\right)^{1/3}
\]
and these tend to \(\infty\) as \(x \to 0\). Notice that the vertical-line test insures that the graph of a function has no vertical secants.

Ex. 3 Does the power function \(px) = x^3\) have a derivative at every point? If so, what is it?
Solution: The familiar graph of \(y = x^3\) suggests that there's a tangent at every point, making the answer to the first question "yes". But to answer the second question (and confirm the tentative "yes") we have to consider limits.
§2.7

We want to know whether for any \( x_0 \) the difference quotient

\[
(7) \quad \frac{p(x) - p(x_0)}{x - x_0} = \frac{x^3 - x_0^3}{x - x_0}
\]

has a limit as \( x \to x_0 \) and we want to find that limit value. Generally with polynomials factorization will get you a first down (sometimes a touchdown). Recall

\[
(8) \quad a^3 - b^3 = (a - b)(a^2 + ab + b^2)
\]

which you can confirm by just multiplying out the right side. With \( a = x \), \( b = x_0 \) this converts (7) into

\[
(9) \quad \frac{p(x) - p(x_0)}{x - x_0} = x^2 + xx_0 + x_0^2
\]

which is a polynomial, consequently (box, p.127) continuous at every point \( x_0 \). That is,

\[
\lim_{x \to x_0} (x^2 + xx_0 + x_0^2) = x_0^2 + x_0x_0 + x_0^2 = 3x_0^2
\]

So we get from (9)

\[
(10) \quad p'(x_0) = \lim_{x \to x_0} \frac{p(x) - p(x_0)}{x - x_0} = 3x_0^2, \text{ for every number } x_0.
\]
There's nothing special about exponent "3" in this example.

**Fact**: For every positive integer \( n \) the \( n \)th power function \( p_n(x) = x^n \) is differentiable, and

\[
p_n'(x_0) = nx_0^{n-1}\quad \text{for every } x_0.
\]

Exactly the same argument works if you first dope out what (8) should look like with "n" in place of "3". It is

\[
(a^n - b^n) = (a-b)(a^{n-1} + a^{n-2} b + \cdots + a b^{n-2} + b^{n-1})
\]

checked by multiplying out its right side and observing massive cancellation. Notice the pattern on the right: powers of \( a \) descend, powers of \( b \) ascend and they always add up to \( n-1 \). For practice, go through the details when \( n = 4 \).

Now, taking \( a = x \), \( b = x_0 \) in (8)n you should be able to imitate the analysis in Ex. 3 to arrive at (10)n. (Exercise!) This formula (the power rule, p. 184) will be needed repeatedly throughout the rest of the course!
**Notation.** There is a popular alternative way to write derivatives. One writes
\[ y = f(x) \quad \text{and} \quad \frac{dy}{dx} = f'(x) \]
(read "dy, dx"). When the function comes as a formula without a name like \( f, g \) or \( p \), for example, just \( x^{17} \), it's convenient to write
\[ \frac{d}{dx} (x^{17}) = 17x^{16} \]

**Looking Ahead.** If \( f(x) \) has a derivative at every point \( x \), then a new function \( f(x) \) is begotten. We could graph it and ask about its tangents and its derivatives. Ivory-tower nerdity?

Hardly! In the example of the displacement function \( d(t) \) we found the velocity function \( v(t) = d'(t) = \text{derivative of displacement.} \) But \( v(t) \) itself is differentiable and its derivative (hence the second derivative of \( d(t) \)) is the instantaneous rate of change of velocity with respect to time, that is, the acceleration
\[ a(t) = v'(t), \]
very important in physics!
§2.8 \textbf{DERIVATIVES}

Reviewing from last lecture:

If \( f \) is a \underline{smooth enough} function and we construct its graph, the \underline{geometric} problem of finding the \underline{tangent lines} to this graph turns out to be a problem with \underline{limits} (of difference quotients), namely (see Fig. 1, p. 159)

slope of tangent line @ point \((a, f(a))\)

\[
= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \text{ if this limit exists.}
\]

\underline{Note}. Here and in the text "\(a\)" is used where "\(x_0\)" was used in the preceding lecture.

This limit depends on \(f\) and on \(a\) (nothing else), so we adopt a symbol for it that features these two things. It is
§2.8

\[ f'(a), \text{ verbalized as} \]

"f prime of a" or "f prime at a". Thus

(1) \[ f'(a) \text{ means } \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \]

(if this limit exists and is finite). Moreover, since the number \( f'(a) \) is derived from \( f \), we refer to it as the derivative of \( f \) at \( a \) and we signal its existence by saying that \( f \) has a derivative at \( a \), or that \( f \) is differentiable at \( a \). Since \( x \to a \) means same as \( x - a \to 0 \), if we denote \( x - a \) by \( h \), then (1) can be written

(2) \[ f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}. \]

\textbf{General Education Note} Mathematicians were the first to use the term "derivative" (18th century). Lately economists have adopted this term for a certain (complex) "financial instrument". If you track down their meaning, you'll find nothing so clear and simple as the mathematicians' definition above.
The notation $f'$ suggests another function, and indeed the definition

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

does give us another function, its value at $a$ being that limit. Remember, part of definition (2) of $f'(a)$ is the assumption that the limit exists, that is, the graph of $f$ has a (non-vertical) tangent at the point $(a, f(a))$. All the graphs we've encountered suggest that for this to happen the graph of $f$ certainly cannot have a break at the point $(a, f(a))$, that is, $f$ should be continuous at $x = a$. It's easy and instructive to confirm that this must be so:

**Important Fact:** If $f$ is differentiable at $a$, then $f$ is continuous at $a$.

What is at issue in continuity is whether

$$\lim_{x \to a} f(x) = f(a)$$
equivalently, whether
\[ \lim_{x \to a} [f(x) - f(a)] = 0. \]

Well, the product limit rule (p. 40) delivers this latter in one shot: Since

\[ f(x) - f(a) = (x-a) \cdot \frac{f(x) - f(a)}{x-a}, \]

that rule gives

\[ \lim_{x \to a} [f(x) - f(a)] = \lim_{x \to a} (x-a) \cdot \lim_{x \to a} \frac{f(x) - f(a)}{x-a} = 0 \cdot f'(a) = 0. \]

But mere continuity of \( f \) at \( a \) does not insure that \( f \) has a derivative at \( a \), as we earlier saw in the example

\[ f(x) = |x|, \]

which is continuous at \( a = 0 \), but has no derivative (tangent) there.

**Note.** Since we’re accustomed to writing the variable (input) in a function (formula) as \( x \), for the new function \( f' \) (read “f prime”) we prefer to write formula (2) as

\[ (3) \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \]
The Derivative as an Aid to Graphing

Suppose function $f$ has domain the (open) interval $(a, b)$ and is differentiable at every $x \in (a, b)$. In graphing $f$ an important feature to clarify is on which subintervals of $(a,b)$ is $f$ increasing, on which is it decreasing?

The derivative answers this nicely. Suppose, say, $a < c < d < b$ & $f \uparrow$ on $(c,d)$ and consider two $x$-values in this subinterval $c < x_0 < x < d$.

$f$ increasing means then that
\[ f(x_0) \leq f(x_1), \text{ so } \]
\[ f(x_1) - f(x_0) \geq 0, \]
\[ \frac{f(x_1) - f(x_0)}{x - x_0} \geq 0 \]

that is, all secant lines inside interval $(c,d)$ have positive slope. Now the limit of positive things is also positive (Squeeze Principle, or Theorem 2, p.110), so from (4) follows...
\[ f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} > 0. \]

In words

If \( f \) is increasing on \((c, d)\), then the tangent line to its graph at each point \((x_0, f(x_0))\) where \(c < x_0 < d\), has positive slope.

More important for graphing \( f \), the converse of (5) holds:

If the tangent line to its graph at each point \((x_0, f(x_0))\) where \(c < x_0 < d\), has positive slope, then \( f \) is increasing on \((c, d)\).

All the graphs we've studied confirm this, but a demonstration that every differentiable function \( f \) indeed satisfies (6) will only come later (p. 296), although Fig. 2 illustrates the key idea: every secant is parallel to some tangent:

\[ \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x_0) > 0 \implies f(x_2) - f(x_1) > 0 \implies f(x_1) < f(x_2), \text{ whenever } c < x_1 < x_2 < d. \text{ That is,} \]
§ 2.9

(6)′ \{ \text{If } f'(x_0) > 0 \text{ for all } x_0 \in (c, d), \}
\text{then } f \uparrow \text{ on interval } (c, d). \]

[Remember, "⇒" means "logically implies"; it is different from the "converges to" arrow →. The former connects statements, the latter connects numbers.]

So if we have a good graph of \( f' \) (which is often a simpler function than \( f \) itself) and we read off the intervals over which its graph is above (below) the \( x \)-axis, these will be the intervals over which \( f \) is increasing (decreasing). Note in the second sidebar on p. 166 \( A, B, C \) should read \( A', B', C' \).

Ex. 1 \( f(x) = x^3 - x \)

built from two power functions, which we saw how to differentiate last time:
\[ f'(x) = 3x^2 - 1. \]
So \( f'(x) > 0 \iff 3x^2 - 1 > 0 \iff x^2 > \frac{1}{3} \iff |x| > \frac{1}{\sqrt{3}} \approx 0.577. \]
In particular, we see that
§ 2.9

\[ f'(x) < 0 \text{ for } 0 < x < \frac{1}{\sqrt{3}} \]
\[ f'(x) > 0 \text{ for } x > \frac{1}{\sqrt{3}} \]
which means that
\[ f \downarrow \text{ on } (0, \frac{1}{\sqrt{3}}) \]
\[ f \uparrow \text{ on } (\frac{1}{\sqrt{3}}, \infty) \]
knowing which lets us sketch the graph of \( f \)
for \( x > 0 \). The left side
of the graph is a freebie, because \( f \) is an odd
fct. (check the criterion: \( f(-x) = -f(x) \)), so its
graph is symmetric to the origin.

Notice that the "arms" of the graph behave
just as the criteria on p. 2 of § 2.6 Lecture
say they should: \text{positive lead coefficient,}
\( (=1) \) and \text{odd-degree} \( (=3) \) polynomial.

\underline{Some Other Derivatives}

At this point we know how to differentiate
any pure power function
\[ px^n = x^n \text{ if } n \text{ is a positive integer.} \]
§2.9

In §2.7 we saw, namely, that
\[ p'(x) = nx^{n-1} \quad \text{for every } x \]
(and used this with \( n = 3 \) and with \( n = 1 \) in preceding example). What about root sets? It turns out they all behave well except at \( x = 0 \): We saw last time that \( f(x) = x^{1/3} \) has no derivative at \( x = 0 \).

Ex. 2 \[ f(x) = x^{1/3} \]
For any \( x_0 \) the difference quotient is
\[ \frac{f(x) - f(x_0)}{x - x_0} = \frac{x^{1/3} - x_0^{1/3}}{x - x_0} \]

Once again use the factorization
\[ a^3 - b^3 = (a - b)(a^2 + ab + b^2) \]
here with \( a = x^{1/3}, b = x_0^{1/3} \). You get
\[ x - x_0 = a^3 - b^3 = (x^{1/3} - x_0^{1/3})(x^{2/3} + x^{1/3}x_0^{1/3} + x_0^{2/3}) \]
whereupon (9) reads
\[ \frac{f(x) - f(x_0)}{x - x_0} = \frac{1}{x^{2/3} + x^{1/3}x_0^{1/3} + x_0^{2/3}} \]
Now let \( x \to x_0 \). The functions \( x^{2/3} \) and \( x^{1/3} \) are each continuous (Theorem 7, p. 129). Consequently

\[
\lim_{x \to x_0} \left( x^{2/3} + x^{1/3} x_0^{1/3} + x_0^{2/3} \right) = x_0^{2/3} + x_0^{1/3} x_0^{1/3} + x_0^{2/3} = 3x_0^{2/3}.
\]

If \( x_0 \neq 0 \), we can put (11) into (10) using the quotient limit law (5, p. 104), to get

\[
\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \frac{1}{3x_0^{2/3}} = \frac{1}{3} x_0^{-2/3}.
\]

In other words, the cube-root function is differentiable and

\[
f'(x_0) = \frac{1}{3} x_0^{-2/3} \quad \text{whenever } x_0 \neq 0.
\]

Do you see how (12) relates to the power law (8)? We deduced (8) for integers \( n \), but (12) is formula (8) with \( n = 1/3 \). This suggests (8) may be valid even when \( n \) is any number at all.

Well, in (8) \( x = 0 \) will be problematic if \( n \) is negative and \( x < 0 \) will be problematic if \( n = 1/2 \), \( 1/4 \), \( 1/8 \), etc. The final power law turns out to be

\[
\{ \begin{align*}
f(x) &= x^n \text{ satisfies } f'(x) = nx^{n-1} \\
&\text{for all } x > 0 \text{ and all powers } n.
\end{align*} \]
\]
Sooner or later we will need to know if the exponential function $e^x$, so pervasive in science & economics, is differentiable (its graph certainly suggests that it is!) and what its derivative is, that is, a formula. The paragraph to follow is the necessary heavy-lifting. But note that this is a very different problem from the power law $(8)$. In $(8)$, the variable is in the base, in $e^x$ the variable is in the exponent.

**A VERY Important Limit**

In Lecture §2.3 p.6 we inferred from the graphs that $e^x > x$ for all $x$. But that’s kind of crude. The graph of $e^x$ tells us a bit more (see Figure 4 below):

(3) $e^x > 1 + x$ for all $x$

and it suggests that the line $y = 1 + x$ is
§2.9

is tangent to the graph at \((0,1)\), that is, the \hspace{1cm} \hspace{1cm} \hspace{1cm} (14) tangent to graph of \(e^x\) at point \((0,1)\) has slope 1.

Let's confirm this (it will unlock many doors!). We need an inequality above \(e^x\) to complement (13). Here's how to get it. If \(x > -1\), so that \(1+x > 0\), (13) can be re-written

\[
\frac{1}{1+x} \geq \frac{1}{e^x} = e^{-x}.
\]

Setting \(x = -x < 1\) this reads

(13)' \hspace{1cm} \frac{1}{1-x} \geq e^x.

We combine (13) and (13)' (changing \(x\) back to \(x\) for uniformity) to learn

\[1+x \leq e^x \leq \frac{1}{1-x} \text{ for all } x < 1.\] (14)

which is a useful inequality to remember!

It entails

\[x \leq e^x - 1 \leq \frac{1}{1-x} - 1 = \frac{x}{1-x} \text{ for all } x \leq 1.\]

If \(x\) is positive, we can divide by it, getting

(15) \hspace{1cm} 1 \leq \frac{e^x - 1}{x} \leq \frac{1}{1-x}, \hspace{1cm} \text{if } 0 < x < 1.
Inequalities (15) set us up to use the squeeze principle: both its left and right sides converge to 1 as \( x \to 0 \). So from (15) follows

\[
\lim_{x \to 0^+} \frac{e^x - 1}{x} = 1.
\]

Just a little more work would show that the same is true if \( x \to 0^- \). Hence we have shown

\[
(16) \quad \lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \frac{e^x - e^0}{x - 0} = 1,
\]

which confirms (14).

**Note:** This is not a proof, since the fundamental inequality (13) on which everything ultimately rests was never proved, but visually inferred from a graph, whose small-scale accuracy when \( x \) gets very near 0 may be questionable. It is a very small step from (16) to the whole story, which is

\[
(17) \quad \lim_{x \to x_0} \frac{e^x - e^{x_0}}{x - x_0} = e^{x_0} \quad \text{for every } x_0.
\]
§2.9

We'll do (17) later, but you might give it a try now, using laws of exponents to write

\[
\frac{e^x - e^{x_0}}{x-x_0} = \frac{e^{x-x_0} - 1}{x-x_0} \cdot e^{x_0}
\]

\[y = e^x\]

\[y = \frac{1}{1-x}\]  
(half of a hyperbola)

\[(0,1)\]  
\[(1,2)\]  
\[(1,e)\]

\[x = 0\]  
\[x = 1\]

\[y = y + 1\]

Figure 4
Illustrating inequality (10) on p. 10