I'll open with a problem we analyzed on pp. 3, 4 of Review III. The ideas in it are important enough to repeat.

**Ex. 1** Without a calculator or tables show that the area under the curve \( y = \frac{4}{1+x^2} \) above the x-axis and between the y-axis and the vertical line \( x=1 \) exceeds 3.

**Solution:** Let \( S \) denote the region described. We'll use a lower Riemann sum with 8 equally spaced points

\[ 0 = x_0 < x_1 < x_2 < \cdots < x_8 = 1 \]

See Fig. 1. The function

\[ f(x) = \frac{4}{1+x^2} \]

is decreasing, so the rectangle \( R_j \)
(2) \( R_j = \{(x,y) : x_{j-1} \leq x \leq x_j \text{ and } 0 \leq y \leq f(x_j)\} \) lies wholly inside the \( j \)th part of the subgraph \( S_j = \{(x,y) : x_{j-1} \leq x \leq x_j \text{ and } 0 \leq y \leq f(x)\} \).

Together the \( S_j \) make up \( S \), so

(3) \( \text{Area}(S) = \sum_{j=1}^{\frac{b}{\delta}} \text{Area}(S_j) \geq \sum_{j=1}^{\frac{b}{\delta}} \text{Area}(R_j) \).

But as we see from (2)

\( \text{Area}(R_j) = (\text{base}) \cdot (\text{height}) = (x_j - x_{j-1}) \cdot f(x_j) = \frac{\delta}{8} f(x_j) \).

Therefore (3) gives

\( \text{Area}(S) \geq \sum_{j=1}^{\frac{b}{\delta}} \frac{\delta}{8} f(x_j) = \frac{1}{8} \sum_{j=1}^{\frac{b}{\delta}} f(x_j) \).

Remembering that \( x_j = j/\delta \) and the formula (1) for \( f \), we then get

\( \text{Area}(S) \geq \frac{1}{8} \cdot \left[ \frac{4}{1 + (\frac{8}{1})^2} + \frac{4}{1 + (\frac{3}{2})^2} + \cdots + \frac{4}{1 + (\frac{8}{5})^2} \right] \)

\( = \frac{1}{2} \left[ \frac{64}{65} + \frac{16}{13} + \frac{64}{73} + \frac{4}{5} + \frac{64}{89} + \frac{16}{25} + \frac{64}{113} + \frac{1}{2} \right] \)

\( = \frac{4890183704}{1622495810} \) (remember, no calculator available!)

\( = 3 + \frac{22696274}{1622495810} > 3.01398 \)

\( > 3 \), as we wished to show.
Ex. 2 You and a friend are calculus-savvy but have no calculator. Curiously he has a book in which is written
“Also he made a molten sea of ten cubits from brim to brim, round in compass, and five cubits the height thereof; and a line of thirty cubits did compass it round about.”
Your friend infers that \( \pi \) (= ratio of circumference to diameter of any circle) must be \( \frac{30}{10} = 3 \). With logic alone can you persuade him otherwise?

Solution: You know that \( (\tan^{-1})'(x) = \frac{1}{1+x^2} \).

Therefore by FTC 2
\[
\int_0^1 \frac{1}{1+x^2} \, dx = \left[ \tan^{-1}(x) \right]_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4} - 0
\]
because \( \tan\left(\frac{\pi}{4}\right) = 1 \) means \( \frac{\pi}{4} = \tan^{-1}(1) \). [See the graph of \( \tan^{-1}(x) \) on p. 74.]

Consequently
\[
\pi = 4 \int_0^1 \frac{1}{1+x^2} \, dx = \int_0^1 \frac{4}{1+x^2} \, dx.
\]
But this integral is the area under the curve \( y = \frac{4}{1+x^2} \) and in Ex. 1 we showed that this exceeds 3.01398. Therefore (5) entails
\[
\pi > 3.
\]
Ex. 3 If you know only that $e < \pi$ and have no calculator, can you determine which of the numbers $e^\pi$, $\pi^e$ is larger?

**Solution:**

Look at the function $f(x) = \frac{\ln x}{x}$ for $x > 1$ graphed in Fig. 2.

$$f'(x) = \frac{x \ln'(x) - x' \ln x}{x^2}$$

$$= \frac{x \cdot \frac{1}{x} - 1 \cdot \ln x}{x^2}$$

$$= \frac{1 - \ln x}{x^2}.$$ 

Therefore $f'(x) < 0$ if $\ln x > 1$, that is, if $x > e$.

In other words, as our graph confirms

(8) $f$ is on interval $(e, \infty)$

Since $e < \pi$, (8) entails $f(e) > f(\pi)$, that is,

\[ \frac{\ln e}{e} > \frac{\ln \pi}{\pi} \]

\[ \pi \ln e > e \ln \pi \]

Using the power rule for logarithms (3. on p. 68) this says

\[ \ln (e^\pi) > \ln (\pi^e), \]
and so since $\ln 1 = 0$, we get finally

$$(9) \quad e^x > \pi e$$

(true to the adage "the power is in the power," i.e., bigger exponent prevails over bigger base).

Next I'll do a sample of the review problems from the end of the syllabus. There are 104 altogether. Do 20/day yourself til Wednesday and you're good to go!

\[ \text{Ex. \#25, p.79. Find } \tan(\arcsin(1/2)). \]

\[ \text{Solution: Write } \theta = \arcsin(1/2). \text{ This is the (radian measure of the)} \text{ angle in } [-\frac{\pi}{2}, \frac{\pi}{2}] \text{ such that } \sin \theta = \frac{1}{2}. \text{ (See p.72). The picture is consequently } \]

\[ \text{Therefore } \tan(\arcsin(1/2)) = \tan \theta = \frac{\frac{1}{2}}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}. \]

\[ \text{Ex. \#27, p.79. Any mass } m_0 \text{ of } \text{^{100}Pd} \text{ will decay in half in four days. That is, each passage of 4 days multiplies the mass present by } \frac{1}{2}. \text{ Now } t \text{ days is } (t/4)-\text{four-day periods, each of which multiplies} \]
by \( \frac{1}{2} \). Consequently, the mass \( m(t) \) present after \( t \) days is

\[ m(t) = m_0 \left( \frac{1}{2} \right)^{t/4} \]

Part (d) asks for the time \( t \) such that \( m(t) = 0.01 \) (gram). Since \( m_0 \) is given to be 1 (gram), we are, according to (4), being asked for the \( t \) such that

\[ 0.01 = m_0 \left( \frac{1}{2} \right)^{t/4} = \frac{1}{2}^{t/4} \]

\[ 0.01 = \left( \frac{1}{2} \right)^{t/4} = \frac{1}{2^{t/4}} \]

\[ 2^{t/4} = 100 = 10^2 \]

\[ \frac{t}{4} \log_2 (2) = 2 \quad \text{(power rule for logarithms)} \]

\[ t = \frac{8}{\log_2 2} \quad \text{(days)} \]

Ex. #15, p. 85. Let \( d \) (miles) denote length of the whole journey. Using

\[ \text{distance} = (\text{rate}) \cdot (\text{time}) \]

\[ \text{time} = \frac{\text{distance}}{\text{rate}} \]

we have

(1) time for first half = \( \frac{d/2 \text{ mi}}{30 \text{ mi/hr}} = \frac{d}{60} \text{ hr.} \)

(2) time for second half = \( \frac{d/2 \text{ mi}}{60 \text{ mi/hr}} = \frac{d}{120} \text{ hr.} \)
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average speed = \frac{\text{total distance}}{\text{total time}} = \frac{d}{\frac{d}{60} + \frac{d}{120}}

= \frac{1}{\frac{1}{60} + \frac{1}{120}} = \frac{1}{\frac{3}{120}} = 40 \text{ (mi/hr)}

Note: \frac{30+60}{2} = 45 \text{ is the average of her two individual speeds, not the average speed over the whole course.}

Ex. #3, p.85. Solve the equation

(0) \quad |2x-1| - |x+5| = 3.

Solution: Following worked Example 2, p.83, four scenarios have to be investigated:

(I) \quad |2x-1| = 2x-1 \quad \text{and} \quad |x+5| = x+5

(II) \quad |2x-1| = 2x-1 \quad \text{and} \quad |x+5| = -(x+5)

(III) \quad |2x-1| = -(2x-1) \quad \text{and} \quad |x+5| = x+5

(IV) \quad |2x-1| = -(2x-1) \quad \text{and} \quad |x+5| = -(x+5).

If (I), then (0) reads

\quad \quad 2x-1 - (x+5) = 3 \Rightarrow

(1) \quad x=9, \text{ an answer compatible with (I). (Check!)}

If (II), then (0) reads

\quad \quad 2x-1 + x+5 = 3 \Rightarrow
(2) \( x = -\frac{1}{3} \), an answer incompatible with (II). (Check!)

If (III), then (0) reads
\[-(2x - 1) - (x + 5) = 3 \Rightarrow \]

(3) \( x = -\frac{7}{3} \), an answer compatible with (III). (Check!)

If (IV), then (0) reads
\[-(2x - 1) + x + 5 = 3 \Rightarrow \]

(4) \( x = 3 \), an answer incompatible with (IV). (Check!)

In summary, from (1) and (3) the solutions of eq. (0) are \( x = 9 \) and \( x = -\frac{7}{4} \).

\[ \text{Ex. #48, p.178. } f(x) = \frac{4-x}{3+x} = \frac{-3-x+7}{3+x} \]
\[ = -\frac{3-x}{3+x} + \frac{7}{3+x} = -1 + \frac{7}{x+3} \quad \text{or get this by long division.} \]

So the graph of \( f(x) \) is down shift by \(-1\) and left shift by \(-3\) of the graph of the “mother” hyperbola \( y = \frac{1}{x} \). See Figures 3 and 4. Note three things from Figure 4:

(i) slope is always negative

(ii) when \( |x| \) is large, slope is nearly 0

(iii) when \( x \) is near \(-3\), slope is nearly \(-\infty\).
Properties (i) – (iii) tell us that the graph of \( y = f(x) \) must look somewhat like Fig. 5.

Ex. #47, p. 178. \( f(x) = \sqrt{3-5x} \)

domain is all \( x \) for which \( 3-5x > 0 \), that is \( x \leq 3/5 \).

If \( x < 3/5 \) and \( h \) is small, then \( x + h < 3/5 \) too, so we may form the difference quotient
**CUMULATIVE REVIEW**

\[
\frac{f(x+h)-f(x)}{h} = \frac{f^2(x+h)-f^2(x)}{h(\frac{f(x+h)+f(x)}{2})} = \frac{3-5(x+h)-(3-5x)}{h(\frac{f(x+h)+f(x)}{2})}
\]

\[= -\frac{5}{f(x+h)+f(x)} \cdot \]

Let \( h \to 0 \) and remember that \( f \) is a **continuous** function, meaning that

\[ f(x+h) \to f(x) \text{ as } h \to 0. \]

Therefore (5) implies

\[ f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = -\frac{5}{f(x)+f(x)} \]

\[= -\frac{5}{2\sqrt{3}-5x}. \]

We can use the chain rule and the power rule of differentiation to confirm (6):

If \( u = 3-5x \), then \( f(x) = u^{\frac{1}{2}} \), so

\[ f'(x) = \frac{du^{\frac{1}{2}}}{dx} = \frac{d}{du} (u^{\frac{1}{2}}) \cdot \frac{du}{dx} = \frac{1}{2} u^{-\frac{1}{2}} (-5) = -\frac{5}{2u^{\frac{1}{2}}} = -\frac{5}{2\sqrt{3}-5x}. \]

**Ex. #31, p. 17**

\[ f(x) = \begin{cases} \sqrt{-x} & \text{if } x < 0 \\ 3-x & \text{if } 0 \leq x < 3 \\ (x-3)^2 & \text{if } 3 < x \end{cases} \]

**Note:** 3 is not in the domain of \( f \).
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\[\lim_{x \to 0^{-}} f(x) = \lim_{x \to 0^{-}} \sqrt{-x} = \sqrt{0} = 0 \neq f(0) \]
\[\lim_{x \to 0^{+}} f(x) = \lim_{x \to 0^{+}} (3-x) = 3 = f(0) \}
\implies \lim_{x \to 0} f(x) \text{ does not exist} \implies f \text{ is discontinuous at } 0.

\[\lim_{x \to 3^{-}} f(x) = \lim_{x \to 3^{-}} (3-x) = 3-3 = 0 \]
\[\lim_{x \to 3^{+}} f(x) = \lim_{x \to 3^{+}} (x-3)^2 = (3-3)^2 = 0 \}
\implies \lim_{x \to 3} f(x) \text{ exists and equals } 0. \text{ Nevertheless, } f \text{ is not continuous at } 3 \text{ because } x=3 \text{ is not a point in the domain of } f. \text{ (Review definition 11, p.124 and the sentence following it.)}

Ex. #10, p.177

\[\lim_{u \to 4^{+}} \frac{4-u}{|4-u|} = \lim_{u \to 4^{+}} \frac{4-u}{u-4} \quad , \text{since } u \to 4^{+} \text{ means } u>4, \text{ so } u-4 > 0 \text{ and } |u-4| = u-4 \]

\[= \lim_{u \to 4^{+}} (-1) \]

\[= -1. \]
Cumulative Review

Ex. #5, p. 208. Displacement from 0 on a horizontal line at time $t$ (sec.) is

$$S(t) = \frac{t}{t^2+1} \quad (t \geq 0)$$

and the positive direction is to the right.

$$v(t) = s'(t) = \frac{(t^2+1) \cdot 1 - t \cdot (2t)}{(t^2+1)^2} = \frac{1-t^2}{(1+t^2)^2}.$$  

$$v(t) \geq 0 \iff 1-t^2 \geq 0 \iff 1 \geq t^2 \iff 1 \geq t.$$  

So for the first second, velocity is positive and the particle is moving right, from $s(0)=0$ to $s(1) = \frac{1}{2}$

a distance $s(1)-s(0)$.

At $t=1$, $v(t) = 0$, the particle is at rest. At every $t > 1$ velocity is negative and the particle is moving leftward from $s(1)$ to $s(t)$, so

a distance $s(1)-s(t)$.

Altogether then if $t > 1$ it covers distance

$$[s(1)-s(0)] + [s(1)-s(t)] = [\frac{1}{2}-0] + [\frac{1}{2}-s(t)]$$

$$= 1-s(t) = \frac{t^2+1}{t^2+1} - \frac{t}{t^2+1} = \frac{t^2+1-t}{t^2+1}.$$  

Note: Integration was unavailable at this point in the book and is not needed.
Ex. (See p.257; a fancier version in the lecture notes for §3.10.) A ladder 12 feet long is flat upright against a vertical wall. Its foot is drawn away from the wall and when 8 feet away its speed is 4 ft/sec. How fast is the top of the ladder descending at that instant?

Solution: Let \( d(t) \) feet be the displacement of the foot from the wall, and \( h(t) \) feet that of the top from the ground at time \( t \) seconds. Then Pythagoras tells us that at each instant \( t \)

\[
(7) \quad 12^2 = d^2(t) + h^2(t).
\]

Differentiate this, using product and chain rules:

\[
0 = 2d(t) \cdot d'(t) + 2h(t) \cdot h'(t)
\]

and solve for what is wanted, namely \( h'(t) \):

\[
h'(t) = \frac{-d(t) \cdot d'(t)}{h(t)}
\]

\[
(8) \quad = \frac{-d(t) \cdot d'(t)}{\sqrt{12^2 - d^2(t)}} \quad \text{by (7)}.
\]
Let $t_0$ denote the time at which the foot is 8 feet from the house. That is,

(9) \quad d(t_0) = 8.

We're given that

(10) \quad d'(t_0) = 4

and asked to find $h'(t_0)$. From (8), (9), (10) follows

$$h'(t_0) = \frac{-8 \cdot 4}{\sqrt{12^2 - 8^2}} = \frac{-8 \cdot 4}{\sqrt{80}} = \frac{-8}{\sqrt{5}} \quad \text{ (ft/sec.)}$$

**Ex. 4** Evaluate the indefinite integral

$$\int (4 + x^{1/3})^{3/2} \, dx.$$

**Solution** (by the method of substitution/change of variable) Introduce

(11) \quad u = 4 + x^{1/3}.

Then

$$x = (u-4)^3$$

$$dx = 3(u-4)^2 \, du.$$

$$\int (4 + x^{1/3})^{3/2} \, dx = \int u^{3/2} \cdot 3(u-4)^2 \, du$$

$$= 3 \int u^{3/2} [u^2 - 8u + 16] \, du.$$
After multiplying this all out each term can easily be integrated by the power rule:

\[
\int (4+x^{1/3})^{3/2} \, dx = 3 \int \left[ u^{7/2} - 8u^{5/2} + 16u^{3/2} \right] \, du
\]

\[
= 3 \left[ \frac{2}{9} u^{9/2} - 8 \cdot \frac{2}{7} u^{7/2} + 16 \cdot \frac{2}{5} u^{5/2} \right]
\]

\[
\left(\text{III}\right) \frac{2}{3} (4+x^{1/3})^{9/2} - \frac{48}{7} (4+x^{1/3})^{7/2} + \frac{96}{5} (4+x^{1/3})^{5/2}
\]

**Note:** It would be unpleasant to check this really does differentiate to give \((4+x^{1/3})^{3/2}\) but it's good practice as part of any review!

**Ex.5** A ball is thrown from a rooftop (nothing is said about velocity or direction or height of the roof!) and strikes the ground 5 seconds later at 100 ft/sec. Find the maximum height which the ball reached.

**Solution:** Let \(s(t)\) feet denote vertical displacement (above ground being considered positive) and \(vt(t)\) feet per second the vertical component of velocity \(t\) seconds after release.
Thus \( s(0) \) is the height of the building. The acceleration \( v'(t) \) is the constant acceleration of gravity, \(-32 \text{ ft/sec}^2\). Therefore according to FTC2

\[
v(t_1)-v(0) = \int_0^{t_1} v' = \int_0^t -32 \, dt = -32t,
\]

(12) \( v(t_1) = v(0) - 32t \).

\[-100 = v(5) = v(0) - 32 \cdot 5 \Rightarrow v(0) = 60 \Rightarrow \]

(13) \( v(t) = 60 - 32t \).

Since \( s' = v \), FTC 2 again delivers

\[
s(t_1)-s(0) = \int_0^{t_1} s' = \int_0^t v = 60t - 16t^2 \quad \text{by (13)},
\]

(14) \( s(t) = s(0) + 60t - 16t^2 = s(0) + 4t(15-4t) \).

\[0 = s(5) = s(0) + 4 \cdot 5 (15-4 \cdot 5) = s(0) + 20 \cdot (-5) \Rightarrow s(0) = 100, \text{ so (14) becomes}
\]

(15) \( s(t) = 100 + 4t(15-4t) \).

At the height of the trajectory \( v(t_1) = 0 \). According to (13) this occurs when

\[t = \frac{60}{32} = \frac{15}{8} \text{ (sec.)}\]

At this time the height of the ball is, according to (15)
Cumulative Review

\[ S\left(\frac{15}{8}\right) = 100 + 4 \cdot \frac{15}{8} \left(15 - 4 \cdot \frac{15}{8}\right) = 100 + \frac{15}{2} \left(15 - \frac{15}{2}\right) \]
\[ = 100 + \left(\frac{15}{2}\right)^2 = 158 \frac{1}{4} \text{ feet}. \]

Ex. 6 Use a tangent line to the graph of the cube-root function to find an approximation \( y_0 \) to \( \sqrt[3]{7} \). Illustrate graphically. Is \( y_0 \) an over- or an under-estimate?

**Solution**: The tangent line to the graph of \( f(x) = \sqrt[3]{x} = x^{1/3} \)

at the point \((x, f(x))\) has slope

\[ f'(x) = \frac{1}{3} x^{-2/3}. \]

The line \( T \) tangent at \((8, f(8)) = (8, 2)\) therefore has slope

\[ f'(8) = \frac{1}{3} \cdot 8^{-2/3} = \frac{1}{3} \left(2^3\right)^{-2/3} = \frac{1}{3} \cdot 2^{-2} = \frac{1}{12}, \]

and an equation for \( T \) in "point-slope" form is

\[ \frac{y-2}{x-8} = \frac{1}{12} \]

(16) \[ y - 2 = \frac{1}{12} (x - 8). \]
The point \((x, y_0)\) on \(S\) with first coordinate \(x\) is close to the point \((x, f(x)) = (7, 3\sqrt{7})\) on the graph of \(f\), and lacking any other knowledge the graph suggests \(y_0\) is the best guess for \(3\sqrt{7}\). According to (16) this \(y_0\) is

\[
y_0 - 2 = \frac{1}{12} (7 - 8) = -\frac{1}{12}
\]

\[
y_0 = 2 - \frac{1}{12} = \frac{23}{12}.
\]

The graph shows that \(y_0 > 3\sqrt{7}\).

With TI-83 we can learn that \(3\sqrt{7} \approx 1.913\), whereas \(y_0 \approx 1.916\), so this crude method actually delivers an approximation accurate to within .0037.

Ex.7 The two numbers \(A = 998^{999}\), \(B = 999^{998}\) both exceed the number of electrons in the known universe and by far the capacity of today’s largest computers. Can you ascertain (and confirm!) which of them is the larger?
Solution: Recall the function used in solving Ex. 2,
\[ f(x) = \frac{\ln x}{x} \]
We showed there that it is decreasing on the interval \((e, \infty)\). Since \(e < 998 < 999\),
f will reverse these relations, giving
\[ f(999) < f(998), \text{ that is,} \]
\[ \frac{\ln 999}{999} < \frac{\ln 998}{998} \]
\[ 998 \ln 999 < 999 \ln 998 \]
\[ \ln(999^{998}) < \ln(998^{999}) \]
that is,
\[ \ln B < \ln A. \]
Then \(B > A\) is not possible because \(\ln\) is an increasing function. We conclude that \(B < A\)
\[ 999^{998} < 998^{999} \]
again the larger exponent trumps.

"Consummatur est."
I enjoyed giving these lectures to you!

Two parting thoughts:

"Ad Astra Per Aspera"

["To the stars through hardship"—motto of the State of Kansas, emblazoned on its flag!]

"Laufet, Brüder, Eure Bahn, Wie ein Held zum Siegen!"

["Run your course, brothers (sisters), Like a hero to victory!"

From F. Schiller’s poem Ode to Joy, sung by a massive choir in the finale of Beethoven’s awesome 9th Symphony."]