Recapitulation: In §5.1 we posed the problem of finding the area of the subgraph of a positive function \( f \), that is, the area of the region above an interval \([a, b]\) and under the graph of \( f \):

1. \( S = \{(x, y) : a \leq x \leq b \ \text{and} \ 0 \leq y \leq f(x)\} \).

The fundamental idea (associated to many names but primarily BERNHARD RIEMANN (1826-1866)—short-lived, like many geniuses) is to subdivide the interval \([a, b]\) by points

2. \( a = x_0 < x_1 < x_2 < \cdots < x_n = b \),

between each adjacent pair of them choose a

3. number \( x_j^* \in [x_{j-1}, x_j] \) for each \( j=1, 2, \ldots, n \),

and look at the rectangles

4. \( [x_{j-1}, x_j] \times [0, f(x_j^*)] = \{(x, y) : x_{j-1} \leq x \leq x_j \ \text{and} \ 0 \leq y \leq f(x_j^*)\} \).

See Figure 1. Each such rectangle has area

5. \( \text{base} \times \text{height} = (x_j - x_{j-1}) \cdot f(x_j^*) \)

Very often we consider
only equi-spaced points in (2), so

\( x_j - x_{j-1} = \frac{b-a}{n} \) (usually denoted \( \Delta x \))

for each \( j = 1, 2, \ldots, n \). We add these up, using

the sigma notation (p. A34), and call the result a **Riemann sum**:

\[
R_n = (\Delta x) \cdot f(x_1^*) + (\Delta x) \cdot f(x_2^*) + \cdots + (\Delta x) \cdot f(x_n^*)
\]

(7) \[ = \Delta x \sum_{j=1}^{n} f(x_j^*) \].

We argued that, at least if \( f \) is a **continuous function**, as the number \( n \) of points \( x_j \) increases and the base \( \Delta x = \frac{b-a}{n} \) narrows, the sum (7) is a better and better approximation to the area of \( S \). In fact,

\[
\lim_{n \to \infty} R_n = \text{area}(S).
\]

(8) See Fig.13, p. 374. This limit does not depend on how the intermediate points are chosen. Often we simply take every \( x_j^* \) to be \( x_j \). Or we can choose \( x_j^* \) a point in \( [x_{j-1}, x_j] \) where \( f(x) \) is smallest. In that case the rectangles (4) lie wholly inside \( S \), \( R_n < \text{area}(S) \) and \( R_n \) is called
§6.1  **Areas Between Curves**

A lower approximating sum (Fig. 9, p. 373), first used by Gaston Darboux, 1842–1917. In this case, as \( n \to \infty \) the rectangles “exhaust” \( S \) from within. This idea goes back to ancient Greek scholars, Archimedes and Eudoxus. On the other hand, we can choose \( x^*_j \) a point in \([x_{j-1}, x_j]\) where \( f(x) \) is largest. In that case, the rectangles altogether contain \( S \), \( \text{area}(S) < R_n \), and \( R_n \) is called an upper approximating sum. (Fig. 8, p. 372).

In an advanced course every mathematical claim is proved, but in this introductory course we are content to give plausibility arguments which show how things work. But even in an advanced course (8) would not be proved. What needs proof and would be proved is the existence of the limit appearing in (8). But we have no real definition of area for irregular regions, only a feeling for it. In such circumstances the
plausibility of (8) induces mathematicians to adopt (8) as the very definition of area (S). Our text is careful about this point; [2], p. 374 is a definition, not a theorem. (Lectures were less forthright on this!)

Now we approach the related problem of the area between two curves, the graphs of two functions \( f \) and \( g \), in the same way. When \( g \) is the constant function 0, that is, the \( x \)-axis, we're back to the original problem. Here

\[ f(x^*) - g(x^*) \]

\[ 0 < a < x_j < x_{j+1} < b \]

Figure 2

\[ f(x^*) - g(x^*) \]

\[ 0 < a < x_j < x_{j+1} < b \]

Figure 3

(9) \( f, g \) are continuous functions with common domain \([a, b]\) and (until further notice)

(10) \( g(x) \leq f(x) \) for each \( x \in [a, b] \).
The role of the subgraph is taken over by the "betweeness" set

\[ B = \{(x,y) : a \leq x \leq b \text{ and } y \text{ between } g(x), f(x)\} \]  

Figures 2 and 3 above are two illustrations. Take a subdivision (2), choose intermediate points \( x_j^* \) as in (3), and form the rectangles

\[ \{(x,y) : x_{j-1} \leq x \leq x_j \text{ and } y \text{ between } g(x_j^*), f(x_j^*)\} \]

having area

\[ \text{(base)} \cdot \text{(height)} = (x_j - x_{j-1}) \cdot (f(x_j^*) - g(x_j^*)) \]

\[ = \Delta x \cdot (f(x_j^*) - g(x_j^*)). \]

Any graphs we can draw make it appear that the sum of these areas, that is

\[ \Delta x \cdot \sum_{j=1}^{n} (f(x_j^*) - g(x_j^*)), \]

approximates the area of \( B \), better and better as \( n \) increases. As in §5.1 this is the basis for defining the area of \( B \):

\[ \text{Area } (B) = \lim_{n \to \infty} \Delta x \sum_{j=1}^{n} (f(x_j^*) - g(x_j^*)). \]

Notice two things: There is no problem if \( g \) dips below the \( x \)-axis (as in Figure 3): the
difference \( f(x_i^*) - g(x_i^*) \) still gives the height of the rectangle (4)*. Figure 2, p. 437 shows this particularly well. Secondly, recall what the function \( f-g \) is:

\[
(f-g)(x) = f(x) - g(x).
\]

Therefore (13), says

\[
(13)_2 \, \text{area} (B) = \lim_{n \to \infty} \Delta x \cdot \sum_{j=1}^{n} (f-g)(x_j^*).
\]

But these are just Riemann sums for the function \( f-g \) and consequently they converge to an integral:

\[
(13)_3 \, \text{area} (B) = \int_{a}^{b} (f-g)
\]

which, because of the additive properties of integrals (box, p. 358), can be written

\[
(13)_4 \, \text{area} (B) = \int_{a}^{b} f - \int_{a}^{b} g.
\]

At least if \( g(x) \geq 0 \) for all \( x \), each of these integrals is itself an area, so (Fig. 4)

\[
(13)_5 \, \text{area} (B) = \boxed{ \text{ } } - \boxed{ \text{ } } = (\text{area under } f) - (\text{area under } g).
\]
A subtle matter is when (10) fails, when the curve $g$ rises \underline{above} \ $f$ in places. See Fig. 9, p. 440. If
\[ f(x) < g(x) \text{ for all } x \in [x_{j-1}, x_j], \text{ then} \]
\[ g(x_j^*) - f(x_j^*) > 0. \]
Now the height of rectangle (4)** is
\[ g(x_j^*) - f(x_j^*) = |f(x_j^*) - g(x_j^*)|. \]
This is also the height of the rectangle in the original situation (Figures 2, 3), since the absolutes change nothing if $g \leq f$. The net effect is that in (12) and in (13), (13), we need to insert absolutes to cover all possible cases. So, for example, (13) reads
\[ \text{area}(B) = \int_{a}^{b} |f-g|. \]
This is the \underline{official definition} of area\( \text{(B)}. \)
See (3), p. 440. It's important to notice that in general (13)** and (13)** are not true unless (10) is in force, that is, unless
\[ g(x) \leq f(x) \text{ for all } x \in [a, b]. \]
Now for some examples:
Ex. 1 (HW #25, p. 442) Describe the region B enclosed by the graphs of

\[ f(x) = \frac{2}{x^2+1} \quad \text{and} \quad g(x) = x^2 \]

and compute its area.

Solution: Naturally we first try to sketch the graphs. Both are symmetric to the y-axis because \( f, g \) are both even functions. So we can find the area for \( x > 0 \) and simply double it. See Fig. 5.

It appears that at some point \( x = b \) the graphs cross and that

\[ g(x) \leq f(x) \quad \text{for all} \quad 0 \leq x \leq b. \]

To find \( b \), solve \( f(b) = g(b) \):

\[
\frac{2}{b^2+1} = b^2
\]

\[ 2 = b^2(b^2+1) = b^4 + b^2 \]

\[ 0 = b^4 + b^2 - 2 = (b^2 - 1)(b^2 + 2). \]

Since \( b^2 + 2 \) cannot be 0 (it's at least 2!), this last equation holds just when

\[ 0 = b^2 - 1, \]  \text{that is,} \]

\[ b = 1. \]
Therefore the "between" region is
\[ B = \{(x,y) : 0 \leq x \leq 1 & g(x) \leq y \leq f(x)\} \]
and its area is
\[ \text{area} (B) = \int_0^1 |f-g| = \int_0^1 (f-g) = \int_0^1 f - \int_0^1 g \]
\[ = [2 \tan^{-1}(x)]_0^1 - \left[ \frac{1}{3} x^3 \right]_0^1 \quad \text{by FTC} 2, \]
\[ f(x) \text{ being } (2 \tan^{-1})'(x), g(x) \text{ being } \frac{d}{dx}(\frac{1}{3} x^3), \]
\[ = 2 \tan^{-1}(1) - \frac{1}{3} = 2 \cdot \frac{\pi}{4} - \frac{1}{3} = \frac{\pi}{2} - \frac{1}{3}. \]
The final answer is twice this, as we noted at the beginning.

Ex. 2 (HW #18, p. 442) Describe the region bounded by the graphs of the equations
(14) \( x = y \) and (15) \( 4x + y^2 = 12 \)
and compute its area.

Solution: Rewriting
(15) \( y^2 = 12 - 4x = 4(3-x) \),
this (square) must always be positive, so on the (15) graph we must have \( 3-x > 0 \),
\[ x \leq 3. \]

To each such \( x \) eq. (15) makes two \( y \)-values

\[ f(x) = \sqrt{4(3-x)} = 2(3-x)^{1/2} \]

\[ -f(x) = -2(3-x)^{1/2}. \]

See Figure 6, which shows that the curve (15) comprised of the two functions \( f \) and \(-f\) is just a parabola with vertex \((3,0)\).

Note for later use

\[ f(x) = \frac{d}{dx} \left[ -\frac{4}{3} (3-x)^{3/2} \right] \] by the chain rule.

The points where (14) and (15) intersect have \( x \)-values satisfying

\[ 12 = 4x + y^2 = 4x + x^2 \]

\[ 0 = x^2 + 4x - 12 = (x-2)(x+6). \]

So these points are \((2,2)\) and \((-6,-6)\).

The enclosed region B is evidently two contiguous "between" sets \( B_1 \) and \( B_2 \), described
§6.1  **AREAS BETWEEN CURVES**

with the help of the **identity function**

\[ g(x) = x = \frac{d}{dx} \left( \frac{1}{2} x^2 \right) \]

as

\[ B_1 = \{(x, y): -2 \leq x \leq 2 \land -f(x) \leq y \leq g(x)\} \]

\[ B_2 = \{(x, y): 2 \leq x \leq 3 \land -f(x) \leq y \leq f(x)\} \]

Therefore

\[ \text{area}(B) = \text{area}(B_1) + \text{area}(B_2) \]

\[ = \int_{-2}^{2} (g - (-f)) + \int_{2}^{3} (f - (-f)) \]

\[ = \int_{-2}^{2} (g + f) + \int_{2}^{3} 2f\]

\[ = \int_{-2}^{2} g + \int_{-2}^{2} f + 2 \int_{2}^{3} f \]

\[ = \left[ \frac{1}{2} x^2 \right]_{-2}^{2} + \left[ -\frac{4}{3} (3-x)^{3/2} \right]_{-2}^{2} + 2 \left[ -\frac{4}{3} (3-x)^{3/2} \right]_{2}^{3} \]

by (16), (17) and FTC 2

\[ = \frac{1}{2} 4 - \frac{1}{2} 36 - \frac{4}{3} + \frac{4}{3} 9^{3/2} + 2 \cdot \frac{4}{3} = 21 \frac{1}{3} . \]