§5.1,5.2  THE AREA PROBLEM & INTEGRATION

Introduction

As you may know, calculus comes in two flavors, differential and integral, each solves a fundamental problem via limits. Differential calculus solves the problem of tangents as limits of secants. Integral calculus solves the problem of areas as limits of rectangles. We've done the former and now turn to the latter. It is amazing enough that we can (after Newton & Leibniz) solve these problems at all, but even more amazing, the two problems turn out to be inverses of one another, because it turns out that antidifferentiation solves the area problem. But to see this requires some work.

General Plan

The problem of finding the area of a region bounded by curves is easy if the region can be divided up into triangles or rectangles. Let's call such a region polygonal.
If the region $S$ is not polygonal, we might try approximating its area by the areas of inscribed polygons $S_n$ and circumscribed polygons $S_n$, then taking a limit as these polygons acquire more and more sides and get closer and closer to $S$. If the areas of the outer $S_n$ and the areas of the inner $s_n$ converge to a common number $A$, that should be the area of $S$, by the squeeze principle, since
\[ \text{area}(s_n) \leq \text{area}(S) \leq \text{area}(S_n). \]

The simplest non-polygon would be a rectangle in which one side is replaced by a curve. In fact, any region can be built up from such special ones, so we study them first.

There's a convenient symbolism, which will prove indispensable, for summing up a list of indexed terms. We write
\[ \sum_{j=m}^{n} t_j \text{ for } t_m + t_{m+1} + \cdots + t_{n-1} + t_n \]
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and read this

"the sum of $t_j$ as $j$ runs from $m$ to $n$.

The character $\Sigma$ is the capital Greek letter sigma, to suggest \textit{sum} — "S" has too many other jobs to be used here. See \textcircled{7}-\textcircled{10}, p.383 for the simple rules for working with this sigma notation.

\textit{Area under a smooth curve}

In what follows we consider a \textbf{continuous} function $f$ with domain the closed interval $[a,b]$ and we suppose $f(x) \geq 0$ for all $x$. The set of points $(x,y)$ in the plane satisfying the inequalities

(1) \hspace{1cm} a \leq x \leq b \text{ and } 0 \leq y \leq f(x)$

is denoted $S$ and called the \textbf{subgraph} of $f$ (a term not used in our text). In words, "$S$ is the region above the $x$-axis and below the graph of $f$.

It's a rectangle with its top replaced by a curve, the graph of $f$. See Figure 2."
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The cross-hatched set is

$$S = \{(x, y) : a \leq x \leq b, 0 \leq y \leq f(x)\}$$

\textbf{Figure 2}

We subdivide the interval \([a, b]\) with the numbers

$$a = x_0 < x_1 < x_2 < \cdots < x_{j-1} < x_j < \cdots < x_{n-1} < x_n = b$$

Usually the distance between any two adjacent numbers \(x_j\) and \(x_{j-1}\) is the same and is denoted

$$\Delta x = x_j - x_{j-1}.$$  

In this case

$$b - a = x_n - x_0 = (x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \cdots + (x_2 - x_1) + (x_1 - x_0)$$

$$= \Delta x + \Delta x + \cdots + \Delta x + \Delta x, \text{ } n \text{ terms}$$

$$= n \cdot \Delta x, \text{ so }$$

$$\Delta x = \frac{b - a}{n}.$$  

But the theory does not require all the sub-intervals \([x_{j-1}, x_j]\) have the same length. The subdivision determines an inner polygon \(S_n\) and
an outer polygon $S_n$, as follows:

First the symbol $[a, \beta] \times [\alpha, \delta]$ denotes the rectangle with horizontal sides $[a, \beta]$ and vertical sides $[\alpha, \delta]$. More precisely

$$[a, \beta] \times [\alpha, \delta] = \{(x, y) : a \leq x \leq \beta, \alpha \leq y \leq \delta\}.$$  

Now for each $i$, let

$$M_j = \max f \text{ on } [x_{i-1}, x_j], \quad m_j = \min f \text{ on } [x_{i-1}, x_j].$$

As Figure 3 shows, the part of $S$ specified by

$$x_{i-1} \leq x \leq x_i \text{ and } 0 \leq y \leq f(x),$$

contains the inner rectangle $[x_{i-1}, x_i] \times [0, m_j]$ and is contained by the outer rectangle $[x_{i-1}, x_i] \times [0, M_j]$.

Statements like $*$ are clearer if we use the standard symbols

$$A \subset B$$

to say that set $A$ is contained in set $B$.

Now let

$$\begin{cases} 
S_n \text{ be the union of all the inner rectangles} \\
S_n \text{ outer}
\end{cases}$$
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The smaller regions

\[ \{(x, y) : x_{j-1} \leq x \leq x_j \quad \& \quad 0 \leq y \leq f(x)\} \]

involved in \( \ast \) altogether comprise \( S \). (See Figures 2, 3.) Therefore from the inclusions described in \( \ast \) and definitions (5) it should be clear that

\[ (6) \quad s_n \subset S \subset s_n. \]

Consequently

\[ (7) \quad \text{area}(s_n) \leq \text{area}(S) \leq \text{area}(s_n). \]

What are these areas? The inner and outer rectangles in \( \ast \) have areas

\[ m_j \cdot (x_j - x_{j-1}) \quad \text{and} \quad M_j \cdot (x_j - x_{j-1}), \]

respectively. Then, using the sigma notation

\[ \sum_{j=1}^{n} m_j \cdot (x_j - x_{j-1}) = \text{area}(s_n), \quad \sum_{j=1}^{n} M_j \cdot (x_j - x_{j-1}) = \text{area}(s_n). \]

Therefore (7) reads

\[ (8) \quad \sum_{j=1}^{n} m_j \cdot (x_j - x_{j-1}) \leq \text{area}(S) \leq \sum_{j=1}^{n} M_j \cdot (x_j - x_{j-1}). \]

Since \( f \) is continuous, all the numbers \( f(x) \) for \( x \) in \([x_{j-1}, x_j]\) are close to \( f(x_j) \) if \( \Delta x = x_j - x_{j-1} \) is small. So both \( m_j \) and \( M_j \) are close to \( f(x_j) \), and
both sums in (8) are close to
\[ \sum_{j=1}^{n} f(x_j) \cdot (x_j - x_{j-1}). \]
This fact together with the inequalities (8) suggests (recall the squeeze principle \[ \text{[3, p.110]} \])
that
\[ \sum_{j=1}^{n} f(x_j) \cdot (x_j - x_{j-1}) \] converges to \( \text{area}(S) \) if \( \Delta x \to 0 \).
In a more advanced course (9) would be proved.

In most cases \( \Delta x = \frac{b-a}{n} \), so \( \Delta x \to 0 \) means \( n \to \infty \) and (9) can be written
\[ \lim_{n \to \infty} \sum_{j=1}^{n} f(x_j) \cdot (x_j - x_{j-1}) = \text{area}(S). \]

All the numbers \( f(x) \) with \( x \) in \([x_{j-1}, x_j]\) are close to each other, so in (9), it is not necessary to pick on the right endpoint \( x_j \). Instead of \( f(x_j) \), we could use \( f(x_{j-1}) \) or \( f(x_j^*) \) for any number \( x_j^* \) in \([x_{j-1}, x_j]\).

That is,
\[ \lim_{n \to \infty} \sum_{j=1}^{n} f(x_j^*) \cdot (x_j - x_{j-1}) = \text{area}(S). \]
Like the symbol \( f' \) for the derivative of \( f \), the symbol for the limit in (9) should show us the
function $f$ involved and the interval $[a, b]$ being partitioned. The symbol is

$$\int_a^b f(x) \, dx$$

for

$$\lim_{n \to \infty} \sum_{j=1}^{n} f(x_j^*) \cdot (x_j - x_{j-1})$$

and it's called the integral of $f$ from $a$ to $b$.

The preceding deliberations can be summarized by

$$\int_a^b f(x) \, dx \text{ [defined in (10)] measures the area of the subgraph $S$ of $f$ for every continuous positive function $f$ with domain $[a, b]$. In fact the limit in (10) exists for every continuous function $f$, whether positive or not. The symbol is thought to have originated as a stylized "S", suggesting sum. The "x" and "$dx$" are unnecessary decorations.}$$

$$\int_a^b f \, dx$$

can be written simply $\int_a^b f$.

The sums that appear in (10) are called Riemann sums, after one of the creators of this theory; the special ones in (8) are called lower and upper (Darboux) sums, after a later French contributor.
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Ex. 1. \( a < b, \ f(x) = e^x \). Find the area under the graph of \( f \) and above the interval \([a, b]\), that is, find \( \int_a^b f = \int_a^b e^x \, dx \). (Cf. Ex. 3, p. 385.)

Solution. Given \( n \), let

\[
\Delta x = \frac{b-a}{n}
\]

and subdivision points

\[ x_j = a + j \cdot \Delta x, \quad j = 0, 1, 2, \ldots, n \]

The associated Riemann sum (using \([\,\text{fig.}\, p.383\,]\) is

\[
\sum_{j=1}^{n} f(x_j) \cdot \Delta x = \Delta x \cdot \sum_{j=1}^{n} e^{x_j} = \Delta x \cdot \sum_{j=1}^{n} e^{a+j \cdot \Delta x}
\]

\[
= \Delta x \cdot \sum_{j=1}^{n} e^{a+j \cdot \Delta x} = \Delta x \left( \sum_{j=1}^{n} (e^{\Delta x})^j \right) \cdot e^{a}
\]

In high-school algebra you probably learned how to sum a geometric progression with common ratio \( r \):

\[
r^1 + r^2 + r^3 + \ldots + r^{n-1} + r^n = \frac{r^{n+1} - r}{r - 1},
\]

but to merely check (15) is easier (exercise!): multiply both sides by \( r - 1 \) and observe massive cancellation on the left.
If we apply (15) to (14), with \( r = e^{\Delta x} \), we get

\[
\sum_{j=1}^{n} f(x_j) \Delta x = \Delta x \cdot \frac{(e^{\Delta x})^{n+1} - e^{\Delta x}}{e^{\Delta x} - 1} \cdot e^a
\]

\[
= \Delta x \cdot \frac{e^{\Delta x} (e^{\Delta x})^{n} - e^{\Delta x}}{e^{\Delta x} - 1} \cdot e^a
\]

\[
= \Delta x \cdot \frac{e^{n\Delta x} - 1}{e^{\Delta x} - 1} \cdot e^{\Delta x} \cdot e^a
\]

\[
= \frac{\Delta x}{e^{\Delta x} - 1} \cdot (e^{b-a} - 1) \cdot e^{\Delta x} \cdot e^a \quad \text{by (13)}
\]

\[
= \frac{1}{e^{\Delta x} - 1} \cdot (e^{b-a} - 1) \cdot e^{\Delta x} \cdot e^a.
\]

Since \( e^0 = 1 \), we spy a difference quotient for the exponential function lurking in (16). The fact that \( \frac{d}{dx} e^x = e^x \) means that

\[
\lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} = \frac{d}{dx} e^x @ x = 0 = e^0 = 1.
\]

Also continuity of the exponential function means that

\[
\lim_{\Delta x \to 0} e^{\Delta x} = e^0 = 1.
\]
Since $\Delta x \to 0$ is the same as $n \to \infty$ (look at (13)), if we use (18) and (17) in (16), we can conclude that

$$\lim_{n \to \infty} \sum_{j=1}^{n} f(x_j) \Delta x = \frac{1}{1}(e^{b-a} - 1).$$

That is,

$$(19) \quad \int_{a}^{b} f = e^b - e^a.$$ 

This solves the problem. But it does more! Since $F(x) = e^x$ is an antiderivative of $f(x) = e^x$ when (19) is written as

$$\int_{a}^{b} f = F(b) - F(a)$$

it hints that maybe

$$(20) \text{integration is the same thing as antidifferentiation.}$$

There is further evidence for this on pp. 376-377, where the book shows that distance traveled can be found by taking the limit of Riemann sums for the velocity function $\frac{d}{dt} = v(t)$. (Please read!) But we saw in Ex 7.8.0.357 how to express the distance $s$ in terms of the
velocity \( v \); we found \( s \) to be an antiderivative of \( v \). So, the
(21) integral of \( v \) is an antiderivative of \( v \).

In fact (20) is universally valid, and we'll confirm it in §5.3.

The process of getting the number \( \int_a^b f \) by subdividing \([a, b]\), forming Riemann sums and taking the limit of them as the number \( n \) of subdivision points increases indefinitely is called integration, and the expression is apt because we're fusing (integrating) all the thin rectangles into the actual area. The limit process makes sense even if \( f(x) \) is sometimes negative. Corresponding to the set \( S \) on p.13 we now have "The region \( S \) between the \( x \)-axis and the graph of \( f \), parts of which will lie below the \( x \)-axis and above the graph of \( f \) [so the term "subgraph" is inapplicable]. The areas of these parts will be measured as negative, because (Figure 5)
if \( f(x) < 0 \) for all \( x \) in some subinterval \([x_{j-1}, x_j]\), then \( m_j = \min f \) and \( M_j = \max f \)
will both be negative, the inner rectangle will be
\([x_{j-1}, x_j] \times [M_j, 0]\)
and its area will be \(-M_j \cdot (x_j - x_{j-1}) > 0\), whereas it is \( M_j \cdot (x_j - x_{j-1}) < 0\) that enters the approximating sum in (8). Similarly the term \( f(x_j^*) \cdot (x_j - x_{j-1}) \)
in the Riemann sums (9) is negative.

This is not a defect of our theory of area because in many applications it makes sense to count area under the \( x \)-axis as negative.

**Properties of the Integral**

On p. 388 the book boxes four basic properties of the integral \( \int_a^b f \) — how it changes when \( f \) changes. Properties 1. and 3. are pretty clear from the area significance of the integral and from looking at how the graph of \( c \cdot f \) relates to the graph of \( f \). Property 4. follows from
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the other three. What is not clear from graphs is Property

2. \(\int_a^b (f+g) = \int_a^b f + \int_a^b g\),

because the graph of \(f+g\) bears no obvious geometric relation to the graphs of \(f\) and \(g\), as Figure 12, p.43 testifies. However in this case, consideration of Riemann sums together with the limit law for sums (cf.1, p.104) makes short work of things:

\[
\int_a^b f + \int_a^b g = \lim_{n \to \infty} \sum_{j=1}^{n} f(x_j) \Delta x + \lim_{n \to \infty} \sum_{j=1}^{n} g(x_j) \Delta x
\]

\[
= \lim_{n \to \infty} \left[ \sum_{j=1}^{n} f(x_j) \Delta x + \sum_{j=1}^{n} g(x_j) \Delta x \right]
\]

\[
= \lim_{n \to \infty} \left[ f(x_1) \Delta x + \cdots + f(x_n) \Delta x + g(x_1) \Delta x + \cdots + g(x_n) \Delta x \right]
\]

\[
= \lim_{n \to \infty} \left[ (f(x_1) + g(x_1)) \Delta x + \cdots + (f(x_n) + g(x_n)) \Delta x \right]
\]

\[
= \int_a^b (f+g). \text{ (Note that \(\int\), p.383 was involved.)}
\]

From these four properties follow almost immediately the comparison properties on p.389, such as, \(f \leq g \Rightarrow \int_a^b f \leq \int_a^b g\).
The last basic property concerns how $\int_a^b f$ changes when $b$ changes; more precisely, we have another addition law:

5. $\int_a^b f = \int_a^c f + \int_c^b f$ whenever $a < c < b$.

When $f$ is positive, the right side is the sum of areas of two contiguous parts of the subgraph, so that's obviously the area of the whole subgraph. See Figure 15, p. 389. If $f$ is not positive we pick a positive number $m > -f$ and look at $F = m + f$. It's a positive function. The case of 5. already settled covers both $m$ and $F$, giving

(22) \[ \int_a^b m = \int_a^c m + \int_c^b m, \]

and

(23) \[ \int_a^b F = \int_a^c F + \int_c^b F. \]

Subtract (22) from (23):

\[ \int_a^b F - \int_a^b m = \int_a^c F - \int_a^c m + \int_c^b F - \int_c^b m \]

or, using property 4.

\[ \int_a^b (F - m) = \int_a^c (F - m) + \int_c^b (F - m). \]

This confirms 5., since $F - m = f$. 

The centerpiece of calculus, the fact that
unites its differential and its integral aspects,
is what the title refers to. We've already glimpsed this in the preceding lecture. It affirms, roughly, that integration is the same as antidifferentiation, or that

(1) integration and differentiation are inverse processes.

Once we make this precise, it's surprisingly easy to confirm. As in the preceding lecture, throughout this one

f is a **continuous** function with domain the interval \([a, b]\)

Given any \(a \leq x \leq b\) we can look at \(f\) as a function just on interval \([a, x]\) and write down its integral

\[
\int_a^x f \,dx \quad \text{or} \quad \int_a^x f(x) \,dx.
\]

If \(f\) is positive, this is an area; see Figure 1. In this way we get a **function of** \(x\), which we regard as a (new) variable. We'll call it \(F(x)\). So
§ 5.3, 5.4  **THE FUNDAMENTAL THEOREM**

(2) \( F(x) = \int_a^x f(x) \, dx \) for \( x \in [a, b] \).

\[ F(x) = \int_a^x f \]

\text{area over } [0, x] \text{ under graph } f

**Figure 1.**

One precise version of (1) is that this function \( F \) is an antiderivative of \( f \). That is,

(1)' \( F \) is differentiable & \( F'(x) = f(x) \) whenever \( a < x < b \).

Let's see why this is so and why it is reasonable.

Let \( h > 0 \) be small, small enough that \( x + h < b \).

Finding \( F'(x) \) involves looking at difference quotients.

(3) \[
\frac{F(x+h) - F(x)}{h} = \frac{1}{h} \left[ \int_a^{x+h} f - \int_a^x f \right] \quad \text{by (2).}
\]

Rule 5, p. 389 in our context says that

\[
\int_a^{x+h} f = \int_a^x f + \int_x^{x+h} f,
\]

so (3) becomes

(4) \[
\frac{F(x+h) - F(x)}{h} = \frac{1}{h} \cdot \int_x^{x+h} f.
\]
In case \( f \) is positive, Figure 2 reminds us, and we saw in detail last lecture, that

\[
\int_{x}^{x+h} f \text{ is the area of the subgraph } S = \{(x, y) : x \leq x \leq x+h, 0 \leq y \leq f(x)\}.
\]

When \( h \) is small this subgraph is almost a rectangle of base \([x, x+h]\) and height \( f(x)\). The area of that rectangle is \( h \cdot f(x)\), so

area of \( S \approx h \cdot f(x)\), that is,

\[
\int_{x}^{x+h} f \approx h \cdot f(x),
\]

(5) \[
\frac{1}{h} \int_{x}^{x+h} f \approx f(x).
\]

Together (4) and (5) strongly suggest that

\[
\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x)
\]

which would confirm (1)'s. If \( f \) is not positive, these convincing graphs are not so convincing (because of negative areas), so we have to work a little harder to confirm (1)', but the basic idea is the same.
First, let's use Rule 1, p. 388 with
\[ c = f(x) \]
to see that
\[
\frac{1}{h} \sum_{x}^{x+h} f(x) - f(x) = \frac{1}{h} \left[ \int_{x}^{x+h} f - hc \right] = \frac{1}{h} \left[ \int_{x}^{x+h} f - \int_{x}^{x+h} c \right]
\]
\[ = \frac{1}{h} \int_{x}^{x+h} (f-c) \quad \text{by Rule 4, p. 388.} \]

Let
\[ M(h) = \max \text{ of } |f(x) - c| \text{ for } x \leq x \leq x + h. \]

Then for all such \(x\)
\[ -M(h) \leq f(x) - c \leq M(h). \]

From comparison property 3, p. 389, these inequalities entail that
\[
-M(h) \cdot h \leq \int_{x}^{x+h} (f-c) \leq M(h) \cdot h
\]

\[ -M(h) \leq \frac{1}{h} \int_{x}^{x+h} (f-c) \leq M(h), \quad \text{that is,} \]
\[ \left| \frac{1}{h} \int_{x}^{x+h} (f-c) \right| \leq M(h). \]
Coupled with (6) this gives
\[ \left| \frac{1}{h} \int_{x}^{x+h} f - f(x) \right| \leq M(h), \]
or, using (4)
\[ (8) \quad \left| \frac{F(x+h) - F(x)}{h} - f(x) \right| \leq M(h). \]

\( f \) being continuous at \( x \) means that
\( |f(x) - f(x)| \) is small if \( x \) is close to \( x \).

Therefore all the \( f(x) \) in \( (7) \) are close to \( f(x) = c \)
if \( h \) is small. More precisely,
\[ \lim_{h \to 0} M(h) = 0. \]

Therefore from \( (8) \) we have (squeeze principle)
\[ \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} - f(x) = 0, \]
achieving our goal \( (1)' \). This is the most important fact in our subject, being called
the Fundamental Theorem of Calculus (= FTC).

Its traditional expression is:

If \( f \) is continuous on interval \([a, b]\), then
\[ (9) \quad \frac{d}{dx} \left( \int_a^x f \right) = f(x) \text{ for all } x \in (a, b). \]

In words, once more,

The integral of \( f \) is an antiderivative of \( f \).
Since we have in principle a way to compute $\int_a^x f$ (namely, as a limit of Riemann sums), we have a way, actually a practical way thanks to computers, to get an antiderivative of any continuous function, although we shouldn't expect it to always be one of handful of elementary functions we're familiar with that have nice recipes (formulas).

We can run this machine in reverse gear: instead of differentiating an integral, what if we integrate a derivative? We get another form of FTC: If $f$ is continuous on $[a, b]$ and $F$ is an antiderivative of $f$, that is, $F' = f$,

then

\[ \int_a^x f = F(x) - F(a) \quad \text{for all } x \in [a, b], \]

which is better remembered as

\[ \int_a^x F' = F(x) - F(a) \quad \text{for all } x \in [a, b]. \]
The Fundamental Theorem

In words (always less precise)

The integral of $F'$ is $F$.

Why is (10) true? $F$ being an antiderivative of $f$ means

$$\frac{d}{dx} F(x) = f(x) \quad \text{for all } x \in (a, b)$$

and by (9) we also have

$$\frac{d}{dx} \left( \int_a^x f \right) = f(x) \quad \text{for all } x \in (a, b).$$

When two functions have the same derivative throughout an interval, they can only differ by a constant, \( \square \), p.294. So there is a number $c$ such that

$$\int_a^x f = F(x) + c \quad \text{for all } x \in [a, b].$$

Take $x=a$ and you get 0 on the left, resulting in $c = -F(a)$, and the preceding equation reads

$$\int_a^x f = F(x) - F(a),$$

just what (10) is claiming.

In (9) and (10) we have the traditional two forms of the FTC. Our text labels them FTC 1 and FTC 2, respectively. The two forms illustrate
§5.3, 5.4  **The Fundamental Theorem**

A pair of inverse operations carried out in opposite orders, analogous to

\[(\sqrt{x})^2 = x \quad \text{and} \quad \sqrt{y^2} = y.\]

In the Hindu religion the god Vishnu can assume numerous incarnations, called avatars. We could say (9) and (10) are two avatars of the FTC.

**Ex. 1** The position or displacement function \(s(t)\) of an object moving on a straight line has derivative \(s'(t) = v(t)\), the velocity. If this is positive for \(t\) between time \(a\) and later time \(b\), the distance covered is

\[s(b) - s(a) = \int_a^b s'(t) \, dt = \int_a^b v(t) \, dt.\]

Formula (10) and Example 1 illustrate that \(\int_a^b F'\) is the net change as \(x\) goes from \(a\) to \(b\) of whatever the function \(F(x)\) is measuring. In Ex. 1 that's displacement from a starting \((t=a)\) position, because \(v > 0\) means the straight-line motion never reverses (and consequently change of displacement = distance covered).
Ex. 2 (cf. p. 409) The velocity function of a particle moving on a straight line is \( v(t) = t^2 - 3 \). How much distance does it cover in the time interval \( 0 \leq t \leq 3 \)?

Solution:

\[ \begin{align*}
\text{Figure 3} & & \text{Figure 4} \\
\end{align*} \]

The graph of \( v \) shows that

\[
(11) \quad v(t) \begin{cases} 
\leq 0 & \text{if } 0 \leq t \leq \sqrt{3} \\
> 0 & \text{if } \sqrt{3} \leq t \leq 3.
\end{cases}
\]

We can't just integrate \( v \) because, as we see on p. 382 [also Figure 3, p. 409], the negative velocities would contribute negatively to the integral \( \int_0^3 v \) would give \( s(3) - s(0) \), the difference in the positions, the displacement from initial position, not the ground covered, what the odometer would show. Rather distance is the
integral of speed, and

speed is the absolute value of velocity.

So we want \( S_0^3 \|v(t)\|. \) From (11) we learn, and in Figure 4 we see, that

\[
|v(t)| = \begin{cases} 
-u(t) & \text{if } 0 \leq t \leq \sqrt{3} \\
v(t) & \text{if } \sqrt{3} \leq t \leq 3 
\end{cases}
\]

Therefore, since

\[
F(t) = \frac{1}{3}t^3 - 3t \text{ is an antiderivative of } u(t),
\]

\[
\int_0^3 |v| = \int_0^{\sqrt{3}} |v| + \int_{\sqrt{3}}^3 |v| \quad \text{by rule 5, p. 389}
\]

\[
= \int_0^{\sqrt{3}} -v + \int_{\sqrt{3}}^3 v = -\int_0^{\sqrt{3}} v + \int_{\sqrt{3}}^3 v
\]

\[
= -[F(\sqrt{3}) - F(0)] + [F(3) - F(\sqrt{3})] \quad \text{by FTC 2}
\]

\[
= F(3) - 2F(\sqrt{3})
\]

\[
= \left( \frac{1}{3}3^3 - 3 \cdot 3 \right) - 2\left( \frac{1}{3}(\sqrt{3})^3 - 3\sqrt{3} \right) = 4\sqrt{3}.
\]

**Notational Matters**

Ex. 2 conceals a notational problem:
§5.3.5.4  The Fundamental Theorem

If we hadn't introduced the name \( F(t) \) for the antiderivative \( \frac{1}{3}t^3 - 3t \) of \( u(t) \), how would we write

\[
\int_a^b u = F(b) - F(a)
\]

as FTC2 requires? Answer: with the symbol

\[
\left[ \frac{1}{3}t^3 - 3t \right]_a^b \text{ meaning } [\frac{1}{3}b^3 - 3b] - [\frac{1}{3}a^3 - 3a].
\]

The function \( F(x) = \int_a^x f \) is a very specific antiderivative of \( f(x) \), according to FTC1. But if \( c \) is any number whatsoever, \( c + \int_a^x f \) is also an antiderivative of \( f(x) \), and besides these there are no others. Thus \( \int_a^x f \) is the unique antiderivative of \( f(x) \) which is 0 at \( x = a \).

Like bad grammar (sic), the naked symbol

\[
\int f(x) \, dx
\]

is the popular way to signal any antiderivative of \( f(x) \), that is, any of the infinitely many functions \( c + \int_a^x f \), without being specific about
the constant $c$. Because of this non-specificity

\[ \int f(x)\,dx \] is called an indefinite integral (of $f$).

See the cautionary (in red!) note on p. 405. A typical example is

\[ \int \ln x\,dx \] signals a function $F(x)$ satisfying $F'(x) = \ln x$.

In this case we can go beyond symbols and exhibit an antiderivative in terms of common functions, namely

\[ F(x) = x \ln x - x. \]

You can easily check that this function does satisfy $F'(x) = \ln x$, but you may (should) ask, how did I come up with it? You'll learn that in Calc. II—something to look forward to!