§4.8 Applications to Business

There's a little technical jargon not internal to mathematics but relevant to business, clearly set out in bold type in this section (pp. 342, 344). You'll need it in the HW and it's probably good to have in your vocabulary. Figure 1, p. 342 is the graph of a cost function $C(x)$, the cost of producing $x$ items (or units). You expect the domain to be $[0, \infty)$, the positive reals, even when the units are discrete, like golf-balls, because if calculus is to be applicable modelling must be via continuous functions. You might also expect $C(0)=0$, but that is not the case in Figure 1. Why? Well, there are costs (like overhead, rent and personnel) even if 0 items are being produced.

Note the book's careful typographic distinction between $C(x)$ and $c(x)$: The cost per unit when $x$ units are produced is $\frac{C(x)}{x}$. This is, of course, another function of $x$. It is denoted $c(x)$ and called the average-cost function.
§4.8 Applications to Business

(1) \[ C(x) = \frac{C(x)}{x} \]

On the other hand, if we look at how the (total) cost changes when production increases (or decreases) by \( \Delta x \), then around production-level \( x \) the cost change per unit is

\[ \frac{C(x+\Delta x) - C(x)}{\Delta x} \]

and the limit of this as \( \Delta x \to 0 \) has the significance of being the rate of change of cost at production-level \( x \). By analogy with distance per unit time as time converges to 0 being instantaneous rate of change of distance with respect to time (=instantaneous speed), we'd like to call the limit

\[ C'(x) = \lim_{\Delta x \to 0} \frac{C(x+\Delta x) - C(x)}{\Delta x} \]

the instantaneous rate of change of cost, but there's no time involved here. Instead the accepted language is

(2) \( C'(x) \) is called the marginal-cost function.
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The production-level \( x \) at which the cost per unit (that is, \( c(x) \)) is minimal occurs either at \( x = 0 \) or at an \( x \) where

\[
(3) \quad c'(x) = 0.
\]

Definition (1) and the quotient rule express

\[
c'(x) = \frac{x C'(x) - C(x)}{x^2}
\]

and consequently the \( x \) where (3) happens satisfy

\[
0 = x C'(x) - C(x), \quad \text{that is,}
\]

\[
(4) \quad C'(x) = \frac{C(x)}{x} = c(x)
\]

In words

The per-unit cost \( c(x) \) is minimal at a production-level \( x \) where \( c(x) \) equals the marginal cost \( C'(x) \).

The book briefly explains why (5) is plausible; see p. 343. But beware: (5) is also true with "minimal" replaced by "maximal", since \( c'(x) = 0 \) signals either relative min. or relative max. To see which, one must look at \( c''(x) \) and use the 2nd derivative test (p. 301).
§4.8 Applications to Business

Worked Example 1, pp. 343–4 illustrates all this with a simple quadratic cost function.

In marketing, as contrasted to manufacturing, emphasis is on the price per unit at production-level \( x \), called simply the \( p(x) \). [also called the demand function — why?]. It's analogous to \( c(x) \). In terms of it the expected revenue at production-level \( x \) would be

\[
(x \text{ units}) \cdot (p(x) \text{ dollars/unit}) = xp(x) \text{ dollars.}
\]

Consequently

\[
(6) \quad R(x) = xp(x)
\]

is called the revenue function and naturally

\[
(7) \quad P(x) = R(x) - C(x)
\]

is the profit function. Again note the type-font distinction between (6) and (7). Just as with \( C(x) \), the derivatives \( R'(x) \), \( P'(x) \) are not called instantaneous rates but marginals.
and the equation

\[ P'(x) = R'(x) - C'(x) \]

following from (7) shows that \( P'(x) = 0 \), that is, production-level \( x \) is a relative min. or a relative max., when

\[ R'(x) = C'(x), \]

that is,

(8) \( P'(x) = 0 \) when marginal revenue = marginal cost.

This is a necessary condition for a relative max. in \( P(x) \), but maybe not sufficient. To be sure such an \( x \) signals a relative max. and not a relative min., we need to know that \( P''(x) < 0 \) (2nd derivative test, p. 301). That is, from (7)'

\[ R''(x) - C''(x) = P''(x) < 0 \]

\[ R''(x) < C''(x). \]

In words, the rate of increase of marginal revenue is less than the rate of increase of marginal cost, at production-level \( x \), if level \( x \) maximizes profit \( P(x) \).

[Do you find this reasonable? The book offers no elucidation.]
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Worked Example 3, p. 346 illustrates all this, but the wording of the solution could be clearer! Maybe this will help:

According to the survey each succeeding group of 20 units sold beyond the initial 200 requires a price decrease of $10 per unit from the original $350 per unit. \( x \) sales means \( \frac{x-200}{20} \) groups of 20 units beyond the initial 200. So for sales level \( x \), the price \( p(x) \) [per unit] has to be

\[
10 \cdot \frac{x-200}{20}
\]

less than the original price of 350. That is,

\[
p(x) = 350 - 10 \cdot \frac{x-200}{20} = 450 - \frac{1}{2}x.
\]

Thereafter the book is clear.

Ex. 1. See #19, p. 347 for the statement.

A somewhat simplistic assumption is made that the demand function (= price function) \( p(x) \) is linear, that is, has the form
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(10) \( p(x) = ax + b, \quad 0 \leq x \leq 55000 \)

for some numbers (coefficients) \( a, b \). Here \( x \) is the number of items produced (= tickets sold). We're told

\[ 10 = p(27000) \quad \text{and} \quad 8 = p(33000) , \]

information that lets us determine \( a, b \) in (10):

\[ 8 - 10 = p(33000) - p(27000) = a \cdot (33000 - 27000) \]

\[ a = \frac{-2}{6000} = \frac{-1}{3000} . \]

Then

\[ 10 = p(27000) = a \cdot 27000 + b = \frac{-1}{3000} \cdot 27000 + b = -9 + b \]

\[ b = 19. \]

Consequently

\( (10) \quad p(x) = \frac{-1}{3000} x + 19 . \)

The revenue is

\[ R(x) = (\text{n.o. of sales})(\text{price per sale}) = xp(x) \]

\[ = x \cdot \left( -\frac{1}{3000} x + 19 \right) = \frac{-1}{3000} x^2 + 19x, \quad 0 \leq x \leq 55000 \]

Its graph is a downward opening parabola which cuts the \( x \)-axis at
\( x = 0 \) and at \( x = 19 \cdot 3000 = 57000 \), as (11) clearly shows. (Fig. 1)
The absolute max. of this parabola occurs at its vertex, midway between the x-intercepts, namely, at $x = 28,500$, which is within the domain of the revenue function $R(x)$, so the max. of $R(x)$ occurs here, and is

$R(28500) = 28500p(28500) = 28500\left(\frac{-1}{3000} 28500 + 19\right)$

$= 2166000$ (dollars).

[As with all quadratic polynomials, no calculus was needed.]

Ex. 2 (#54, p. 339 — negotiating a corner) This classic problem was mentioned on p. 5 of the §3.10 lecture, along with a reference (p. 6) where variants of it can be found in the literature. We have here a simplified and concrete numerical example: Hallways of width 6 and 9 intersect at a right angle. A pipe of length $l$ (measured in whatever units 6 and 9 are) passes around the corner. What is the longest $l$ can be? Idealize the pipe to a 1-dimensional line segment.
Solution: As formulated here, the problem is underdetermined, because surely the height of the ceiling matters: if there is no ceiling at all, any length (oriented vertically) will go around. But generally if the ceiling has height $h$ and the longest pipe that will go around horizontally has length $l$, then the longest pipe that will go around at all (tilted to reduce its effective length) has length $\sqrt{h^2 + l^2}$, as Fig. 2 illustrates. So it's enough to solve the problem for horizontal passage (which is actually the way the book formulated it), that is, reduce to a 2-dimensional problem. Consider Figure 3 in which the pipe which at angle $\theta$ just touches both walls has length $L(\theta)$. If $l$ is to negotiate the corner, it clearly must satisfy

$$l \leq L(\theta) \text{ for every } 0 \leq \theta \leq \frac{\pi}{2}.$$
This means that the maximum length \( l \) that makes it completely around the corner is the minimum value of the \( L \) function. Let's find that. From Figure 3

\[(13) \quad L(\theta) = L_1(\theta) + L_2(\theta)\]

and from the very definitions of sine, cosine

\[
\frac{6}{L_2(\theta)} = \sin \theta \quad \text{and} \quad \frac{9}{L_1(\theta)} = \cos \theta,
\]

so (13) says

\[(14) \quad L(\theta) = \frac{9}{\cos \theta} + \frac{6}{\sin \theta}.
\]

At the two endpoints this (always positive) function is infinite. Therefore its absolute minimum occurs in the open interval \((0, \frac{\pi}{2})\), is a relative min. and a critical point, and therefore satisfies

\[0 = L'(\theta) = \frac{9}{\cos^2 \theta} \cdot \sin \theta - \frac{6}{\sin^2 \theta} \cdot \cos \theta\]

\[
\frac{9}{\cos^2 \theta} \cdot \sin \theta = \frac{6}{\sin^2 \theta} \cdot \cos \theta
\]

\[9 \sin^3 \theta = 6 \cos^3 \theta
\]

\[\tan^3 \theta = \frac{5 \sin^3 \theta}{\cos^3 \theta} = \frac{\frac{6}{9}}{\frac{2}{3}} = \frac{2}{3}
\]

\[(15) \quad \tan \theta = \left(\frac{2}{3}\right)^{\frac{1}{3}} = \frac{2^{\frac{1}{3}}}{3^{\frac{1}{3}}}.
\]
From Figure 4 (Pythagoras) we see that \( \cos \theta, \sin \theta \) correspond to (15), letting us finally compute (14):

\[
\cos \theta = \frac{2^{1/3}}{\sqrt{2^{2/3} + 3^{2/3}}}, \quad \sin \theta = \frac{2^{1/3}}{\sqrt{2^{2/3} + 3^{2/3}}},
\]

\[
L(\theta) = 9\sqrt{2^{2/3} + 3^{2/3}} \frac{3^{1/3}}{2^{1/3}} + 6\sqrt{2^{2/3} + 3^{2/3}} \frac{2^{1/3}}{3^{1/3}}
\]

\[
= (9 \cdot 3^{1/3} + 6 \cdot 2^{1/3}) \sqrt{2^{2/3} + 3^{2/3}}
\]

\[
\approx 21.0705.
\]

This is the max. possible length. According to (15) it touches the walls just when

\[
\theta = \arctan \left( \frac{2^{2/3}}{3^{1/3}} \right) \approx 0.718 \text{ radians} \approx 41.14 \text{ degrees}.
\]

**Ex. 3** (= HW #21, p.347) The set-up is like worked Example 3, p.346. Just adapt the comment on it made above to the present numbers. The unit price \( p(x) \) at sales-level \( x \) (= the demand function) in this case will then be

\[
p(x) = 450 - 10 \cdot \frac{(x-1000)}{100} = 550 - \frac{1}{10} x.
\]

The worked example asked us to maximize the
revenue function

\[(17) \quad R(x) = xp(x) = x \cdot (550 - \frac{1}{10}x) \]

and part (b) of the present problem asks for that too—proceed as before.

In addition, here we're given a cost (of production) function

\[(18) \quad C(x) = 68000 + 150x \]

and asked to maximize the profit

\[ P(x) = R(x) - C(x) \]
\[ = x \cdot (550 - \frac{1}{10}x) - (68000 + 150x) \quad \text{by (17), (18)} \]
\[ = -\frac{1}{10}x^2 + 400x - 68000. \]  

Once again, the graph is an open-down parabola (Fig. 5), so its max. occurs at its vertex \((x_0, P(x_0))\), which we know how to find, either by calculus or by algebra (=completing the square in (19)). For example, from

\[ 0 = P'(x_0) = 2 \cdot \frac{-1}{10}x_0 + 400 \]

we find \(x_0 = 2000\). Hence profit is maximized when 2000 units are sold, at a price per unit of

\[ p(x_0) = 550 - \frac{1}{10}x_0 = 550 - 200 = 350 \quad \text{(dollars)}, \]

that is, with a $100 rebate off the original $450 price.
The section title is almost self-explanatory: the function $F$ that solves a problem may not be known, but we may be able to figure out what $F'$ is (often true in physics, where rates of change can be measured); say it's the specific function $f$:

\[(1) \quad F' = f.\]

Recovering $F$ from $F'$ is called \underline{antidifferentiation} and $F$ is called an \underline{antiderivative} of $f$. If we think of differentiation as an operator on functions, we might write $F'$ as $DF$. Then (1) reads

\[(2) \quad DF = F' = f.\]

It's natural to designate the $F$ that satisfies this equation by

\[(3) \quad F = D^{-1}f.\]

In this sense antidifferentiation is an inverse operation, the \underline{inverse of differentiation}. The problem with this symbolism is that there are many $F$ which satisfy (2); because if $F$ is one, then so are $F+16$, $F+341$, $F-193$, $F+\pi$, etc.
Which one of these is $D^{-1}f$ supposed to be specifying? So our book avoids (3)—although you will encounter it in differential equations courses. Actually, when you have one antiderivative $F$ for $f$ on the interval $I$, every other antiderivative $G$ is simply $F+c$ for some constant $c$. That is,

Any two antiderivatives of $f$ on an interval differ only by a constant.

For if $F'=f$ and $G'=f$, then $(F-G)'=F'-G'=0$, so $F-G$ is constant according to (5), p. 293. Consequently

If two antiderivatives of $f$ agree at one point in an interval, then they are identical.

Most functions $f$ that come up in calculus and applications have antiderivatives, although they are often not expressible in convenient formulas. The big theoretical result here (which we'll study soon) is a mechanism in principle for finding an antiderivative of
any continuous function. [The procedure is called integration.] Thus, for example, the function $f(x) = \tan x \cdot \ln x$ is the derivative of some function $F(x)$, but $F(x)$ is not in the basic catalog of functions we're familiar with.

What we can do is make a table of all the functions $F$ we know how to differentiate, together with their derivatives $F' = f$. Listing the $f$'s first gives us a table of antiderivatives. See [2], p. 354.

For example, for any integer $k$

$$(4) \quad \frac{d}{dx} (x^k) = kx^{k-1} \quad (x \neq 0 \text{ if } k < 0)$$

says that $x^k$ is an antiderivative of $kx^{k-1}$. If we ask for an antiderivative for $x^n$ where $n$ is an integer, and if $n \neq -1$, we can get one by applying (4) with $k = n + 1$, giving

$$\frac{d}{dx} (x^{n+1}) = (n+1)x^n, \quad \text{that is,}$$

$$\frac{d}{dx} \left( \frac{1}{n+1} x^{n+1} \right) = x^n, \quad \text{or in words}$$

$$(5) \quad \frac{1}{n+1} x^{n+1} \text{ is an antiderivative of } x^n \text{ if } n \neq -1.$$
What if \( n = -1 \)? We know

\[ (6) \quad \ln'(x) = \frac{1}{x} \quad \text{if } x > 0, \]

which we can write as

\[ (7) \quad \frac{d}{dx} (\ln |x|) = \frac{1}{x} \quad \text{if } x > 0. \]

Now form (7) is also true for \( x < 0 \). Because if \( x < 0 \) the chain rule says

\[
\frac{d}{dx} (\ln |x|) = \ln' |x| \cdot \frac{d}{dx} |x| = \frac{1}{|x|} \cdot \frac{d}{dx} |x| \\
= \frac{1}{|x|} \cdot \frac{d}{dx} (-x) \\
= \frac{1}{|x|} \cdot (-1) = \frac{1}{-|x|} = \frac{1}{x} \quad \text{using the fact \( |x| = -x \) when } x < 0.
\]

In summary

\[ (8) \quad \frac{d}{dx} \ln |x| = \frac{1}{x} \quad \text{for all } x \neq 0. \]

The last two entries in table \( \text{TABLE 21, p.354} \) say in effect that

\[ (\tan^{-1})'(x) = \frac{1}{1 + x^2} \quad \text{for all } x \]

\[ (\sin^{-1})'(x) = \frac{1}{\sqrt{1 - x^2}} \quad \text{for all } x \in (-1, 1). \]

The source of these (implicit differentiation) is pp. 232-233. More novel is the 9th entry in the table:
§4.10 ANTIDERIVATIVES & ANTIDIFFERENTIATION

(9) \( \sec x \) is an antiderivative of \( \sec x \tan x \).

To see that this is so, differentiate \( \sec x \):

\[
\sec'(x) = \left(\frac{1}{\cos x}\right)'(x) = \frac{-1}{\cos^2 x} \cdot \cos'(x) \quad \text{chain rule}
\]

\[
= \frac{-1}{\cos^2 x} \cdot (-\sin x) = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}
\]

\[
= \sec x \cdot \tan x.
\]

Formula (8) is behind some antiderivatives like (9) that we could scarcely have guessed, such as

(10) \( \ln |\sec x| \) is an antiderivative of \( \tan x \), \( x \in (-\frac{\pi}{2}, \frac{\pi}{2}) \).

Later we'll learn how (11) could be discovered. But checking its truth is easy; we just see whether \( \ln |\sec x| \) really has \( \tan x \) as derivative. Since \( \cos x \neq 0 \) for \( x \in (-\frac{\pi}{2}, \frac{\pi}{2}) \)—draw the graph—we know that

\[ \sec x = \frac{1}{\cos x} \] makes sense

and by (8)

\[
\frac{d}{dx} \ln |\sec x| = \frac{1}{\sec x} \cdot \frac{d}{dx} \sec x \quad \text{(chain rule)}
\]

\[
= \frac{1}{\sec x} \cdot \sec x \tan x \quad \text{by (10)}
\]

\[
= \tan x, \text{ confirming (11)}.
\]
§4.10 Antiderivatives & Antidifferentiation

In the same way you can check the even more exotic fact that

(12) $\ln|\sec x + \tan x|$ is an antiderivative of $\sec x$, $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

Actually, the only restriction needed on $x$ for (11) and (12) to hold is that $x$ not be an odd integer multiple of $\frac{\pi}{2}$, such multiples being where $\cos x = 0$ and $\sec x$, $\tan x$ are not even defined.

Finding Antiderivatives Graphically

The book discusses two methods. The first we've already seen (essentially) in §4.3, esp. HW #6 and #8 there.

Ex. 1 Figure 1 is the graph of a function $f(x)$. From it construct the graph of a plausible antiderivative $F$ having $F(0) = 2$.

Solution: If an antiderivative of $f$ means that

(1) $F'(x) = f(x)$ for all $x$. 
Figure 1: graph of $f$ ($= F'$)

Figure 2: general shape of an antiderivative $F$ of $f$, having $F(0) = 2$. 
The steps leading to Figure 2 are:

1. \( f = F' < 0 \) in interval \((0, 1)\) \(\Rightarrow\) \( F \downarrow \) in \((0, 1)\).
2. \( f = F' > 0 \) in interval \((1, 5)\) \(\Rightarrow\) \( F \uparrow \) in \((1, 5)\).
3. \( f'(\text{the slope of } f) = F'' \) changes from positive to negative @ \( x = 3 \) and \( F''(3) = f'(3) = 0 \) \(\Rightarrow\) \((3, F(3))\) is an inflection point of graph of \( F \), with \( F \) concave up on interval \((1, 3)\), concave down on interval \((3, 5)\) — concavity test, p. 300.
4. \( f = F' < 0 \) in interval \((5, \infty)\) \(\Rightarrow\) \( F \downarrow \) in \((5, \infty)\).
5. \( f'(\text{the slope of } f) = F'' \) changes from negative to positive @ \( x = 7 \) \(\Rightarrow\) \( F \) concave down on interval \((5, 7)\), concave up on interval \((7, \infty)\).
6. \( 0 = \lim_{x \to \infty} f(x) = \lim_{x \to \infty} F'(x) \Rightarrow\) graph of \( F \) getting ever flatter as \( x \to \infty \). But this does not necessarily mean that \( F(x) \) has a horizontal asymptote. [For example, \( F(x) = \ln x \) has \( F'(x) = f(x) = \frac{1}{x} \to 0 \) but we know that \( F(x) \) grows (very slowly, it's true) indefinitely large as \( x \to \infty \).]
The other graphical technique is to realize that the number \( f(x_0) \), which we may have thanks to either a graph or a table, is the slope of the \( F \)-graph at the point \((x_0, F(x_0))\). That is, when the graph of \( F(x) \) crosses the vertical line \( x = x_0 \), it does so with slope \( f(x_0) \). This information can be captured graphically by short line segments up and down the line \( x = x_0 \), each of slope \( f(x_0) \). If we do this for many \( x_0 \)'s in the interval \([a, b]\) on which \( f \) is defined, we get what's called the direction field determined by \( f \) (Figure 5, p.356). The graph of any antiderivative \( F \) of \( f \) must at each point have these (tangent) directions, that is, pass through the field parallel to all these direction indicators (Figure 6, p.356). Starting at any height \( c \) over \( a \), there is only one such way through the field, that is, just one antiderivative \( F \),
corresponding to the fact that antiderivatives of \( f \) which are equal at \( a \) are identical [see (4) above]. This once merely theoretical method is now very practical and widely used in applications.

**Galileo Again**

In Ex. 3 of the §3.3 lecture we discussed the trajectory of a thrown object. (Actually, “trajectory” is from the Latin word “jaceere” = to throw.) We used the formula

\[
(13) \quad d = 16t^2
\]

for the distance (in feet) fallen from rest by an object after elapsed time \( t \) (seconds). Galileo discovered this experimentally and less than a century later Newton “explained” it. He observed that all falling bodies experience constant acceleration of 32 ft/sec per second. That is, with the customary convention
that up is positive, the velocity increases by 
-32 ft/sec every second. Newton was inventing
the calculus about the same time and in
terms of it this fact can be recorded as

\[(14) \quad v'(t) = -32.\]

[By the way Newton, and many people today
when doing mechanics, wrote \( v(t) \) for \( v'(t) \)
that is, generally used \( \text{prime} \) where our text uses
\( \text{prime} \) \] Fact (14) seems more basic than
fact (13) and entails (13). How? Suppose a
body falls from

initial height \( h_0 \) with initial velocity \( v_0 \).

Let \( h(t) \) be its height at time \( t \) (before
reaching earth). Then the velocity is

\[(15) \quad v(t) = h'(t).\]

Since \( \frac{d}{dt} (-32t) = -32, \) (14) and (15) say that
there is a constant \( c \) such that

\[ v(t) = (-32t) = c \]

Then \( v_0 = v(0) + 32 \cdot 0 = c, \)
\[ (16) \quad \mathbf{v}(t) = v_0 - 32t. \]

Similarly,
\[ \frac{d}{dt} (v_0 t - 16t^2) = v_0 - 32t = \mathbf{v}(t) = h'(t), \]
so (16) says there is a constant \( k \) such that
\[ h(t) = k + v_0 t - 16t^2 \]
\[ h_0 = h(0) = k + v_0 \cdot 0 - 16 \cdot 0^2 = k, \]
so
\[ (17) \quad h(t) = h_0 + v_0 t - 16t^2. \]

Of course, if \( v_0 = 0 \) (falling from rest) this says distance fallen in time \( t = h_0 - h(t) = 16t^2 \), confirming Galileo's discovery (13).

HW #76, p.360 is another problem involving constant acceleration motion in a straight line. Ex.7, p.357 shows how the same arguments as above work to give the displacement function \( h(t) \) when the acceleration is not constant but linear, that is,
\[ h''(t) = v'(t) = a(t) = \alpha t + \beta \]
for some constants \( \alpha, \beta \) and the motion is along a straight line (called rectilinear).