Yours truly graded p.3 on 8 of the 16 sections. Disconcertingly often the following errors (inter 2lii ) appeared:

1. \[ \tan^{-1}(x) = \frac{1}{1+x^2} \]

If you look at Figure 25, p. 74, you see that 
\[ \tan^{-1}(1) = \frac{\pi}{4} \]
(just because \( \tan \left( \frac{\pi}{4} \right) = 1 \)). But the right-hand side of (1) when \( x = 1 \) is \( \frac{1}{2} \), not \( \frac{\pi}{4} \). So
(1) is false.

What is true is

1'. \[ (\tan^{-1})'(x) = \frac{1}{1+x^2}, \text{ for all } x. \]

Some (fortunately, not many) are unaware that for functions \( f \)
f^{-1} is not the same as \( \frac{1}{f} \).

See the red caution p.65. In particular, 
\[ \tan^{-1} \neq \frac{1}{\tan} \] (which is cot).

Also, after arriving in problem 4(b) at
\[ \ln y = \tan(t) \ln 10, \]
too many differentiated (chain rule on left) to get
(2) \( \frac{1}{y} \cdot y' = (\ln y)' = \tan'(t) \ln 10 + \tan(t) \cdot \frac{1}{10} \),

whereas the correct right-hand side is

\( \frac{1}{y} \cdot y' = (\ln y)' = \tan'(t) \ln 10 = \sec^2(t) \ln 10 \),

because for any number \( c \) and function \( f \)

(3) \( (cf)' = cf' \).

If you forgot this basic fact (box, p.197), use the product rule

(4) \( (cf)' = cf' + c'f \)

and remember (again, box, p.197) that constant functions all have derivative 0, so \( c' = 0 \) and (4) is same as (3). The unwanted term in (2) may arise from misunderstanding basic symbolism:

\[ \ln'(10) = \frac{1}{10} \quad \text{but} \quad (\ln 10)' = 0, \]

for the first is \( \ln'(x) = \frac{1}{x} \) at \( x = 10 \), whereas the second is \( c' \) for the constant \( c = \ln 10 \).

Mathematics is no more forgiving of our using symbols carelessly than are computers.
§4.5 \underline{Curve Sketching}

We've already experienced how to use the various theoretical results, like the Intermediate-value and Mean-value theorems, the second-derivative concavity criterion, etc. to work up accurate graphs of fairly complicated functions. In this section we plan to systematize this and do more examples.

On p.316 the book answers the question which all students ask: Why is all this necessary? If you give me the function \( f(x) \), can't I just plot a few hundred (!) points and draw a smooth curve through them to see what the graph of \( f \) looks like? The example given on p.316 shows just one of the pitfalls (or peaks?) of such a bare-hands, "no-theory" approach. Here is another:

\[ f(x) = x^{10} - 10^x. \]

Plot (or ask TI-83 to plot) some points.
\[ f(-2) = (-2)^{10} - 10^{-2} = 1024 - \frac{1}{100} = 1023.99 \]
\[ f(-1) = 1 - 10^{-1} = 0.9 \]
\[ f(0) = 0 - 10^0 = -1 \]
\[ f(1) = 1 - 10^1 = -9 \]
\[ f(2) = 2^{10} - 10^2 = 924 \]
\[ f(3) = 5849 \]
\[ f(5) = 9665625 \]
\[ f(7) = 272475249 \]

This suggests the graph rises indefinitely (Figure 1) as \(1x1\) ↑. But lo and behold (2) \( f(10) = 0 \).

"Theory" would have put us on guard to this because of the fact that (3) as \(x\to0\) the exponential function overwhelms every power function.

This fact was hinted at on p.13 of §4.1 lecture and a special case of it is Example 2, p.309; it follows without too much work from the basic inequality
\[ e^x > x \text{ for all } x. \]

In the present situation (3) means
\[ \lim_{x \to \infty} \frac{10^x}{x^{10}} = \infty \]
and consequently from (1)
\[ \lim_{x \to \infty} f(x) = \lim_{x \to \infty} x^{10} \left(1 - \frac{10^x}{x^{10}}\right) = \infty \cdot (-\infty) = -\infty. \]

So the graph of \( f \) is very far from looking like Figure 1, and a zero on the positive \( x \)-axis was predicted by the Intermediate Value Theorem (\cite{10}, p.131).

As to systematizing, the book lists 7 matters that should be addressed in graphing any function \( f \). See next page. In the preceding several lectures we have worked up graphs of functions, accounting for all these features, but not in the methodical check-list fashion of an aircraft pilot. The book gives 5 good examples of this methodical procedure on pp. 319-323, and we'll do a couple more here.
§4.5 \textbf{CURVE SKETCHING}

The check-list:

A. Find the \underline{domain} of \textit{f}: Over what \underline{part} of the \underline{x-axis} will there be \underline{graph}?

B. Find \underline{all} \underline{x-} and \underline{y-} \underline{intercepts}. Of course, then
    \underline{can} be \underline{at most one} of the latter. (Why?)

C. \underline{Check for symmetry} (even or odd \textit{fct.})
    as well as \underline{periodicity}: a number \textit{p} (such
    as \underline{2\pi} for the \textit{trig.} \textit{functions}), \underline{satisfying}
    \( f(x+p) = f(x) \) \underline{for all} \textit{x}.

D. \underline{Asymptotes}.

E. \underline{Intervals of increase} and \underline{intervals of decrease}.

F. Local \underline{maxima} and local \underline{minima}.

G. \underline{Intervals of concavity}, direction of concavity, \underline{inflection points}.

\textbf{D needs elaboration.} Up til now we have seen \underline{horizontal asymptotes}, which are \underline{lines}
\( y = L \) that the graph hugs when \( |x| \) is big!
(4) \( \lim_{x \to -\infty} f(x) = L \) and/or \( \lim_{x \to \infty} f(x) = L \)

("long-term" behavior), and vertical asymptotes, which are lines \( x = c \) that the graph hugs when \( x \) is near \( c \):

(5) \( \lim_{x \to c^-} f(x) = \infty \) and/or \( \lim_{x \to c^+} f(x) = \infty \)

("blow-up" usually caused by a denominator that becomes \( 0 \) when \( x = c \)).

But besides (4) another long-term behavior is possible and **all** polynomials display it, that is, "arms":

(6) \( \lim_{x \to -\infty} f(x) = \pm \infty \), \( \lim_{x \to \infty} f(x) = \pm \infty \).

In one special case the arms are almost straight lines. Consider a rational function

(7) \[ f(x) = \frac{N(x)}{D(x)} , \]

\( N(x), D(x) \) being polynomials. Suppose

(8) degree of \( N(x) \) = 1 + degree of \( D(x) \).

Remember the **Division Algorithm** (= Long Division)
from algebra; it lets us carry out the division in (7), giving a quotient and a remainder. Precisely it says that there are two other polynomials \( Q(x) \) [called the quotient] and \( R(x) \) [called the remainder] satisfying

\[
N(x) = Q(x) D(x) + R(x)
\]

and the important restriction that

\[
\text{degree of } R(x) < \text{degree of } D(x).
\]

When polynomials multiply, their degrees add (why?), so from (9)

\[
\text{degree of } N(x) = \text{degree of } Q(x) + \text{degree of } D(x)
\]

Therefore (8) tells us

\[
\text{degree of } Q(x) = 1,
\]

that is,

\[
Q(x) = ax + b
\]

for some numbers \( a, b \). Hence (9) gives us

\[
F(x) = \frac{N(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)} = ax + b + \frac{R(x)}{D(x)}.
\]

According to (10) the highest power of \( x \) in \( D(x) \) exceeds the highest power of \( x \) in \( R(x) \). Since long-term behavior depends only on the highest power
§4.5 Curve Sketching

\[ \lim_{|x| \to \infty} \frac{R(x)}{D(x)} = 0. \]

(See the calculation atop p.139.) So we learn from (11) that

\[ \lim_{|x| \to \infty} [ax+b-F(x)] = 0. \]

This says that the graph of \( F(x) \) is getting ever closer to the straight line \( y=ax+b \), it has that line as oblique or slanted asymptote. See Figure 2.

Ex. 2 Graph the rational function

\[ F(x) = \frac{2x^2+3x}{x+1} \]

Solution. The domain is all \( x \neq -1 \), and the line \( x=-1 \) is a vertical asymptote. Long division leads to

\[
\begin{align*}
2x+1 \\
\frac{x+1}{2x^2+3x} \\
\frac{2x+2}{2x^2+2x} \\
\frac{x}{x+1} \\
-1
\end{align*}
\]

\[ Q(x) = 2x+1, \quad R(x) = -1. \quad \text{Thus} \]

(13) \[ F(x) = \frac{2x^2 + 3x}{x+1} = 2x + 1 + \frac{-1}{x+1} \]

\[ 2x + 1 - F(x) = \frac{1}{x+1} \]

\[ \begin{cases} < 0 \text{ if } x < -1 \\ > 0 \text{ if } x > -1. \end{cases} \]

This says

\[ F(x) \text{ is} \begin{cases} \text{above } 2x + 1 & \text{if } x < -1 \\ \text{below } 2x + 1 & \text{if } x > -1. \end{cases} \]

but in either case

\[ \lim_{x \to \infty} \left[ 2x + 1 - F(x) \right] = \lim_{x \to \infty} \frac{1}{x+1} = 0 \]

and the line

\[ y = 2x + 1 \]

is an oblique asymptote.

See Figure 3. That graph makes it appear \( F \) is everywhere increasing.

To confirm this, look at

\[ F'(x) = (13) 2 + \frac{1}{(x+1)^2} > 0 \]

for all \( x \neq -1 \). Therefore indeed

(14) \( F \uparrow \text{ on } (-\infty, -1) \) and \( F \uparrow \text{ on } (-1, \infty). \)
Moreover,
\[ F''(x) = \frac{-2}{(x+1)^3} \begin{cases} > 0 & \text{if } x < -1 \\ < 0 & \text{if } x > -1, \end{cases} \]
again confirming what Figure 3 suggests:

(5) \( F \) concave-up on \((-\infty, -1)\), concave-down on \((-1, \infty)\).

Note that although direction of concavity changes at \( x = -1 \), there is no inflection point (Why?) See second box, p. 300.

The (always unique) y-intercept is \((0, F(0)) = (0, 0)\).

The x-intercepts are the solutions of \(0 = F(x) = \frac{x(2x+3)}{x+1}\), namely, where
\[ 0 = x(2x+3), \text{ that is,} \]
\[ x = 0 \text{ and } x = -3/2. \]

There is no symmetry, no extrema and no periodicity.

HW #37, p. 323 is a clone of worked Example 4, p. 320. So let's move on:

Ex. 3 (HW #42, p. 323) Sketch the graph, covering check-list items A.–G., of the function
\[ f(x) = e^{2x} - e^x = e^x(e^x - 1). \]

Solution. The domain is all \( x \). Note
\[
e^{x} - 1 \begin{cases} < 0 & \text{if } x < 0 \\ > 0 & \text{if } x > 0, \end{cases}
\]
whereas \( e^x > 0 \) for all \( x \); so the factored version of (16) shows

\[ f(x) \begin{cases} < 0 & \text{if } x < 0 \\ > 0 & \text{if } x > 0. \end{cases} \]

There are no vertical asymptotes, but
\[
\lim_{x \to -\infty} e^x = 0 \text{ and } \lim_{x \to 0} e^x = \infty, \text{ so}
\]

\[ \lim_{x \to -\infty} f(x) = 0, \]
making the negative \( x \)-axis a horizontal asymptote, and the factored version of (16) shows

\[ \lim_{x \to 0} f(x) = \infty, \infty = \infty, \]
so there is a raised right arm. Figure 4.

\[ f'(x) = 2e^{2x} - e^x = 2e^x(e^x - 1/2), \text{ and} \]
\[ e^x - 1/2 < 0 \iff e^x < 1/2 \iff x < \ln(1/2) = -\ln 2 \ (\approx -0.7). \]

So, as before, the factored (20) shows
\[
f(x) \begin{cases} < 0 & \text{if } x < -\ln 2 \\ > 0 & \text{if } x > -\ln 2. \end{cases}
\]
Hence (box, p. 296)

(21) \( f \downarrow \) on \((-\infty, -\ln 2)\), \( f \uparrow \) on \((-\ln 2, \infty)\)

and at \( x = -\ln 2 \) there is a relative (and absolute) minimum. The minimum value is

\[
f(-\ln 2) = e^{-2\ln 2} - e^{-\ln 2} = \frac{1}{(e^{\ln 2})^2} - \frac{1}{e^{\ln 2}} = \frac{1}{2^2} - \frac{1}{2} = \frac{1}{4}.
\]

Back to (20)

\[
f''(x) = 4e^{2x} - e^x = 4e^x(e^x - 1/4).
\]

As before, this leads to

\[
f''(x) \begin{cases} < 0 & \text{if } x < -\ln 4 = -2\ln 2 \\ > 0 & \text{if } x > -\ln 2. \end{cases}
\]

Consequently (box p. 300)

(22) \( f \) is

\[
\begin{cases} \text{concave-down on } (-\infty, -2\ln 2) \\ \text{concave-up on } (-2\ln 2, \infty). \end{cases}
\]

Since direction of concavity changes at \( x = -2\ln 2 \)

\((-2\ln 2, f(-2\ln 2)) = (-2\ln 2, -3/16)\)

is an inflection point.

Check-list complete; ready for take-off, that is, graphing. See Figure 5, which incorporates all we learned above.

![Figure 5](image-url)
Exhortation: Those fond of calculators (and all engineering majors) should look at the §4.6 that we're skipping. It shows how to use theoretical knowledge we've acquired to enhance and make precise the graphs which calculators generate. Very useful ideas, illustrated with excellent examples and graphics.

The "optimization" in the section title is optimization of some real-world variable which is governed by some function law that lets calculus be brought into play. So word problems (as in §3.10) will again be the setting. Review the principles for setting them up and solving them on pp. 80, 81 and the four general points suggested on pp. 10, 12 of the §3.10 lecture, for example, creating symbolic names for the things given verbally, and keeping dimensions in the set-up equations. A shorter "To-Do" list on
§4.7  Optimization Problems

pp. 331–332 summarizes all this.

Look at the book’s excellent worked examples on pp. 332–336.

Remark 1 The book’s Example 1, p. 332, fencing a pasture along a river, is really a simple algebra problem, requiring no calculus. After the set-up phase we find the area function

(1) \( A(x) = 2400x - 2x^2 = 2x(1200-x), \quad 0 \leq x \leq 1200 \)

is to be maximized. That is, we seek numbers \( x_0 \in [0, 1200] \) satisfying

(2) \( A(x) \leq A(x_0) \) for all \( x \in [0, 1200] \).

The graph of the quadratic function \( A(x) \) is a parabola, which opens down due to the negative coefficient on \( x^2 \). From (1) it is clear that its \( x \)-intercepts are \( x = 0 \) and \( x = 1200 \), so the graph is Figure 1.

Obviously there is exactly one \( x_0 \), doing job (2); it is
\( x_0 = 600, \) and for it

\[
A(x_0) = 2 \cdot 600 (1200 - 600) = 2 \cdot 600^2 = 720,000.
\]

Another purely algebraic way to see this is by “completing the square” (which, incidentally, is how the “quadratic formula” is found):

\[
A(x) = -2(x^2 - 1200x) = -2(x^2 - 1200x + \left(\frac{1200}{2}\right)^2 - \left(\frac{1200}{2}\right)^2)
\]

\[
= -2(x^2 - 1200x + 600^2) + 2(600)^2
\]

\[
= -2(x - 600)^2 + 720,000
\]

\( \leq 720,000, \) with equality to this maximum value just when \( x = 600. \)

Remark 2 Example 5, p. 376 can also be solved without calculus (and more simply!) if one expresses the rectangular area to be maximized in terms of a more appropriate (angle) variable, instead of the more immediate height of the rectangle. See the book’s second solution.
Ex. 1 (#16, p. 337) Find the point(s) on the line \( L \) defined by \( 6x + 2y = 9 \) that is closest to the point \((-3, 1)\).

**Solution.** As the book noted in its Example 3, minimizing distance is the same as minimizing squared distance (why?) and the latter is computationally easier because of the absence of square roots. Re-write the equation of line \( L \) as

\[
(3) \quad y = -3x + 9/2.
\]

According to Pythagoras the distance from \((-3, 1)\) to any point \((x, y)\) is

\[
\sqrt{(x-(-3))^2 + (y-1)^2} = \sqrt{(x+3)^2 + (y-1)^2}.
\]

If that point is on line \( L \), then \( y = -3x + 4.5 \) by (3), so this distance is

\[
\sqrt{(x+3)^2 + (3.5 - 3x)^2}.
\]
Therefore we seek the number(s) $x$ that minimizes the square of this distance, that is, minimizes the function

\[(4) \quad f(x) = (x+3)^2 + (3.5-3x)^2 \]
\[= x^2 + 6x + 9 + (3.5)^2 - 2(3.5)x + 9x^2 \]
\[= 10x^2 - 15x + \frac{85}{4} = 10\left(x^2 - \frac{3}{2}x\right) + \frac{85}{4} \]
\[= 10\left(x^2 - \frac{3}{2}x + \left(\frac{3}{4}\right)^2\right) - 10\left(\frac{3}{4}\right)^2 + \frac{85}{4} \]

\[(5) \quad = 10\left(x - \frac{3}{4}\right)^2 + \frac{141}{8}. \]

This is the same idea as in Remark 1 earlier. From (5) it is clear that

\[f(x) \geq \frac{141}{8} \quad \text{for all } x, \text{ with equality just at } x = \frac{3}{4}. \]

The point sought is therefore the point on $L$ with first coordinate $x = 3/4$, namely

\[(6) \quad \left(x, \frac{9-6x}{2}\right) = \left(\frac{3}{4}, \frac{9-6\left(\frac{3}{4}\right)}{2}\right) = (\frac{3}{4}, \frac{3}{4}). \]

Solution 2. To minimize $f$ over all $x \in \mathbb{R}$, differentiate (4) and find where $f'(x) = 0$:

\[0 = f'(x) = 2(x+3) + 2(3.5-3x)(-3). \]

Note chain rule was used to differentiate the second square term. Simplifying
\[ 0 = 2x + 6 + (7 - 6x)(-3) = 20x - 15 \]
\[ x = \frac{3}{4}, \text{ as found earlier, and we get} \]
\[ \text{point } \theta \text{ just as before.} \]

**Solution 3.** The nearest point on \( L \) lies on the line \( L' \) through \((-3, 1)\) which is perpendicular to \( L \). The slopes of perpendicular lines are related by
\[ -1 = (\text{slope } L) \cdot (\text{slope } L') = -3 \cdot \text{slope } L', \]
\[ \text{slope } L' = \frac{1}{3}. \]

Therefore our point \((x, y)\) satisfies ("point-slope" form of a line)
\[ \frac{y-1}{x-(-3)} = \frac{1}{3} \tag{7} \]

and of course, being a point on \( L \), it satisfies (3) as well. Substituting (6) into (7) again leads to \( x = \frac{3}{4} \).

**Ex. 2** (#31, p.337) Designate by \( s \) (meters) the length destined to be bent into a square (of side-length \( \frac{3}{4} \)). Then \( 10 - s \) is destined to be bent into an equilateral triangle, of side-length \( \frac{10 - s}{3} \).
The area of an equilateral triangle of side-length \( l \) is (see Figure 3)
\[
\frac{1}{2} h \cdot l = \frac{1}{2} l \sin \left( \frac{\pi}{3} \right) l = \frac{1}{2} \cdot \frac{\sqrt{3}}{2} l^2
\]
In our case \( l = \frac{10-s}{3} \), so the combined area of the two figures is
\[
A(s) = \left( \frac{5}{4} \right)^2 + \frac{\sqrt{3}}{4} \left( \frac{10-s}{3} \right)^2
\]
\[
= \frac{1}{16} s^2 + \frac{\sqrt{3}}{4} \left( \frac{10}{3} \right)^2 - \frac{20}{9} s + \left( \frac{5}{3} \right)^2
\]
\[
= \left( \frac{1}{16} + \frac{\sqrt{3}}{4} \cdot \frac{1}{9} \right) s^2 - \frac{5\sqrt{3}}{9} s + \frac{25\sqrt{3}}{9},
\]
an opening-up parabola which (8) shows lies wholly above the \( s \)-axis.
(See Figure 4.) Its maximum over \([0,10]\) will certainly occur at one endpoint, and the minimum will occur at the vertex if that falls in the interval \([0,10]\), otherwise at the other endpoint. By now we know two ways to finish off the problem: by algebra, that is, “complete the square” in (9), or by calculus, that is, find where \( A'(s) = 0 \). Enough said.
Ex. 3 (#56, p.340)

The depth of the gutter is (Figure 5) $10 \sin \theta$ (cm) and the cross-sectional area, which alone determines its carrying capacity, is unchanged if it's re-configured by moving interior triangle T to exterior space S. The cross-sectional area of the rectangle thus formed is

$$A(\theta) = \text{(base)} \cdot \text{(height)} = (10 + 10 \cos \theta) \cdot 10 \sin \theta$$

(10) $= 100 (1 + \cos \theta) \sin \theta$

and we need to maximize this function over $\theta \in [0, \pi]$. Using the product rule on (10)

$$A'(\theta) = 100 (-\sin \theta) \sin \theta + 100 (1 + \cos \theta) \cos \theta$$

$= 100 (-\sin^2 \theta + \cos \theta + \cos^2 \theta)$

$= 100 (\cos^2 \theta - 1 + \cos \theta + \cos^2 \theta)$

$= 100 (2 \cos^2 \theta + \cos \theta - 1)$

$= 100 (2 \cos \theta - 1)(\cos \theta + 1)$.

Therefore
Optimization Problems

\[ 0 = A'(\theta) \iff \cos \theta = \frac{1}{2} \text{ or } \cos \theta = -1 \]
\[ \iff \theta = \frac{\pi}{3} \text{ or } \theta = \pi. \]

There is some \( \theta \in [0, 2\pi] \) where \( A(\theta) \) is maximal (thanks to the Extreme-Value Theorem [3], p. 281) and this \( \theta \) is certainly not an endpoint, since \( A(0) = A(\pi) = 0 \). So this \( \theta \) lies in the open interval \((0, \pi)\) and is a local maximum. But then according to [6] and [7], p. 283 we must have \( A'(\theta) = 0 \), which by (11) means that
\[ \theta = \frac{\pi}{3} \]
is the maximizing angle.

Ex. 4 (Snell's Law: #51, p. 339) Light has speed \( v_j \) (m/sec.) in medium \( M_j \) \((j=1,2)\) and in passing between them always minimizes the time (Principle of Least Action in physics).

Show that the angle \( \theta_1 \) of exit from \( M_1 \) and the angle \( \theta_2 \) of entry into \( M_2 \) (see Figure 6) satisfy

\[ \frac{\sin \theta_1}{\sin \theta_2} = \frac{v_1}{v_2}. \]
Solution. Choose a coordinate system whose \( x \)-axis is the interface and whose \( y \)-axis passes through the light source \((0, y_1)\) in \(M_1\). Let \((x_2, y_2)\) be the image point in \(M_2\), and let \((x, 0)\) denote the transition point. The time in \(M_1\) is \(\frac{\text{distance}}{\text{speed}} = \frac{\sqrt{x^2+y_1^2}}{v_1}\).

The time in \(M_2\) is \(\frac{\text{distance}}{\text{speed}} = \frac{\sqrt{(x_2-x)^2+y_2^2}}{v_2}\).

So the total time for the trip (function of \(x\)) is

\[
T(x) = \frac{1}{v_1} \sqrt{x^2+y_1^2} + \frac{1}{v_2} \sqrt{(x_2-x)^2+y_2^2}
\]

and the \(x \in \mathbb{R}\) that minimize this are sought. At any extreme point \(T'(x) = 0\). But an easy computation with (13) gives

\[
T'(x) = \frac{1}{v_1} \frac{x}{\sqrt{x^2+y_1^2}} - \frac{1}{v_2} \frac{x_2-x}{\sqrt{(x_2-x)^2+y_2^2}}
\]
Therefore

\[ T'(x) = 0 \iff \frac{1}{v_1} \frac{x}{\sqrt{x^2+y_1^2}} = \frac{1}{v_2} \frac{x_2-x}{\sqrt{(x_2-x)^2+y_2^2}} \]

But Figure 6 clearly shows that

\[ \frac{x}{\sqrt{x^2+y_1^2}} = \sin \theta_1 \quad \text{and} \quad \frac{x_2-x}{\sqrt{(x_2-x)^2+y_2^2}} = \sin \theta_2. \]

Consequently (14) says

\[ T'(x) = 0 \iff \frac{1}{v_1} \sin \theta_1 = \frac{1}{v_2} \sin \theta_2. \]

This confirms (12) if the extreme point \( x \) is a point of minimum \( T \) and not of maximum \( T \). But it is obvious from (13) that

\[ \lim_{x \to \infty} T(x) = 0 \]

so \( T(x) \) has no maximum, so the extremum found must be a minimum value of \( T \). Consequently, from (15) we’re entitled to infer Snell’s Law (12).