Section 4.6 / Growth & Decay

Exponential Functions

**Exponential Growth Model**

\[ P(t) = P_0 e^{kt} \]

- \( t \): time
- \( k \): exponential growth rate (\( k > 0 \))
- \( P_0 \): initial amount
- \( e \): Euler Number \( 2.718... \)

Examples: Investments
Population growth

**Exponential Decay**

\[ P(t) = P_0 e^{kt}, \quad k < 0 \]

Examples: Half-life problems
 Carbon-dating
Ex. 11 Suppose that in a certain country, the initial population is 2 million people. Also suppose the exponential growth rate is \( k = 0.014 \).

a) Find an exponential growth model that corresponds to this situation.

**Answer:** \( P(t) = P_0 e^{kt} \)

\[
P(t) = 2 e^{0.014t}
\]

where \( P(t) \) is in millions.

b) Assume \( t = 0 \) corresponds to year 2000, what was the population in 2005?

**Answer:** \( t = 5 \) goes with 2005

\[
P(5) = 2 e^{0.014(5)} = \frac{2.145}{1} \text{ million people}
\]

c) How long will it take for the initial population to double?

**Answer:** \( P(t) = 2 e^{0.014t} = 4 \) \( \Rightarrow \) \( 2 e^{0.014t} \) Solve for \( t \).
\[ 2 = e^{0.014t} \]

Convert to logarithmic form (use definition)

\[ \ln 2 = 0.014t \]

\[ \frac{\ln 2}{0.014} = t \]

\[ t \approx 50 \text{ yrs} \]
Ex2) Suppose that in a certain town, in the year 1980, there were 2000 teachers and in 1990, there were 2103 teachers.

a) Assuming exponential growth, find the growth rate $k$.

**Answer:** $P(t) = P_0 e^{kt}$

$$2103 = 2000 e^{10k}$$

$$\frac{2103}{2000} = e^{10k} \quad \text{Exponential Equation}$$

$$\ln \left( \frac{2103}{2000} \right) = 10k \quad \text{Convert to Logarithmic Form}$$

$$\frac{\ln \left( \frac{2103}{2000} \right)}{10} = k$$

$$k \approx .005$$

b) What is the growth model?

**Answer** $P(t) = 2000 e^{.005t}$

c) How many teachers will there be in 2020 (assuming the exponential model holds)
Answer: \[ t=0 \rightarrow 1980 \]
\[ t=40 \rightarrow 2020 \]

\[ P(40) = 2000(e^{0.05 \times 40}) \]
\[ \approx 2443 \text{ teachers} \]
Exponential Decay

The \( \frac{1}{2} \) life of carbon-14 is 5,750 years.

(a) Assuming exponential decay, find the value of \( k \) that corresponds to this situation.

Answer: \[ P(t) = P_0 e^{kt} \]

\[ \frac{1}{2} P_0 = P_0 e^{k(5750)} \]

\[ \frac{1}{2} = e^{5750k} \] (Exponential Equation)

Convert to logarithmic form

\[ \ln \left( \frac{1}{2} \right) = 5750k \]

\[ \ln \left( \frac{1}{2} \right) = k \]

\[ \frac{1}{5750} \]

\[ k \approx -0.00012 \]

(b) What is the model that goes with this situation?

Answer: \[ P(t) = P_0 e^{-0.00012t} \]
e) Suppose an ancient wooden statue is found to have lost 35% of Carbon-14. How old is the artifact?

Answer: 
\[ P(t) = P_0 e^{-0.00012t} \]

\[ 0.65 P_0 = P_0 e^{-0.00012t} \]

\[ 0.65 = e^{-0.00012t} \]

\[ t = \frac{\ln(0.65)}{-0.00012} = 3590 \text{ yrs old} \]
Ex) Continuous Compounding

\[ P(t) = P_0 e^{rt} \]

\[ P(t) = P \left(1 + \frac{r}{n}\right)^{nt} \]

- \( r \): rate of interest
- \( t \): time in yrs
- \( P_0 \): initial investment value

Ex) Suppose that an initial investment, under continuous compounding, after 5 years, grew to $12,000 under an annual 7%. What was the interest rate of initial investment value?

Answer: \[ P(t) = P_0 e^{rt} \]

\[ 12,000 = P_0 e^{0.07 \cdot 5} \]

\[ 12,000 = P_0 e^{0.35} \] — linear equation

\[ \frac{12,000}{e^{0.35}} = P_0 \]

\[ P_0 \approx \$8456.26 \]