QUADRATIC FUNCTIONS

PROTOTYPE:

\[ f(x) = ax^2 + bx + c. \]  \hspace{1cm} (1)

The leading coefficient \( a \neq 0 \) is called the \textit{shape parameter}.

SHAPE-VERTEX FORMULA

One can write any quadratic function (1) as

\[ f(x) = a(x - h)^2 + k, \]  \hspace{1cm} (Shape-Vertex Formula)

where \( h = -\frac{b}{2a} \) and \( k = f(h) = c - \frac{b^2}{4a} \).

EXAMPLE 1. \( f(x) = 2x^2 - 8x + 5 \).

To derive the Shape-Vertex Formula for \( f(x) \) we first identify the coefficients:

\( a = 2, \quad b = -8, \quad c = -1 \).

With these identifications we have:

\[ h = -\frac{b}{2a} = -\frac{(-8)}{2 \cdot (2)} = \frac{8}{4} = 2; \]
\[ k = f(h) = f(2) = 2(2)^2 - 8(2) + 5 = -3, \]

so the Shape-Vertex Formula for \( f(x) \) is:

\[ f(x) = 2(x - 2)^2 - 3 \]
FACTS ABOUT THE GRAPH:

A. The graph has same shape as the graph of $ax^2$, but shifted. The shifting is determined by the numbers $h$ and $k$ that appear in the Shape-Vertex Formula.

We illustrate this fact with Example 1 above. In that example we started with the function $f(x) = 2x^2 - 8x + 5$, and we found the Shape-Vertex Formula to be

$$f(x) = 2(x - 2)^2 - 3.$$  

By the above Fact, we then know that the graph of $f(x)$ is the same as the graph of $y = 2x^2$, but shifted 2 units to the right, and 3 units down.

The graph of $f(x)$ is shown in red, while the graph of $y = 2x^2$ is shown in blue.
B. The shape of the graph of \( f(x) = ax^2 + bx + c \) is called a \textit{parabola}. The parabola opens upward or downward, depending on the sign of the leading coefficient \( a \), as shown below.

\[
\begin{array}{c}
\text{V} \\
a > 0
\end{array}
\quad
\begin{array}{c}
\text{V} \\
a < 0
\end{array}
\]

\textbf{THE VERTEX.} The “tip” of the parabola, marked by \( V \) in the above pictures, is called the \textit{vertex}. Its coordinates are the numbers \((h, k)\), given in the \textit{Shape-Vertex Formula}. The vertical line through the vertex is an \textit{axis of symmetry} for the parabola.

The vertex is a “turning point” (a point where the graph changes direction). Moreover:

- if \( a > 0 \), then the vertex is a \textit{minimum} point;
- if \( a < 0 \), then the vertex is a \textit{maximum} point.

The intervals of monotonicity (where the function is increasing or decreasing) are \((-\infty, h)\) and \((h, \infty)\).
GRAPHING AND ANALYZING THE FUNCTION

Use the following steps when dealing with a quadratic function

\[ f(x) = ax^2 + bx + c. \]

**Step 1.** Find the *y-intercept* \( f(0) \).

**Step 2.** Find the *x-intercept(s)*, by solving the equation \( f(x) = 0 \).

**Step 3.** Find the coordinates of the vertex:

\[ x_V = h = \frac{-b}{2a}; \quad y_V = k = f(h) = c - \frac{b^2}{4a}. \]

**Step 4.** Draw the graph. (Use the information from Steps 1-3.)

**Step 5.** Analyze the graph and extract information about the function.

- specify whether the vertex is a maximum or a minimum point;
- indicate the intervals where the function is increasing or decreasing.
EXAMPLE. Graph and analyze \( f(x) = -x^2 - 2x + 3 \).

Solution: Step 1. The \( y \)-intercept is \( y = f(0) = -(0)^2 - 2(0) + 3 = 3 \).

Step 2. The \( x \)-intercept(s) are found by solving the equation:
\[-x^2 - 2x + 3 = 0.\]
Using the Quadratic Formula, the solutions are
\[x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot (-1) \cdot (3)}}{2 \cdot (-1)} = \frac{2 \pm 4}{-2},\]
so there are two \( x \)-intercepts: \( x_1 = -3 \) and \( x_2 = 1 \).

Step 3. We find the numbers \( h, k \):
\[h = -\frac{b}{2a} = -\frac{(-2)}{2 \cdot (-1)} = -1,\]
\[k = f(h) = f(-1) = -(1)^2 - 2(-1) + 3 = 4,\]
so the vertex is the point \((-1, 4)\).

Step 4. The graph is shown on the right.

Step 5. The vertex \((-1, 4)\) is a maximum point. The function \( f(x) \) is:
- increasing on \((-\infty, -1)\);
- decreasing on \((-1, \infty)\).
FINDING THE FUNCTION, GIVEN THE VERTEX

When the vertex of the graph is given, we proceed as follows.

**Step 1.** Replace \(h, k\) in the Shape-Vertex Forumula

\[
f(x) = a(x - h)^2 + k,
\]

so that we get a “preliminary” form of the function:

\[
y = a(x - \#)^2 + \#.
\]

(Here it is understood that \(\#\) mean concrete numbers.

**Step 2.** Replace \(x\) and \(y\) by the coordinates of the other point given, so that now we would get something like:

\[
\# = a(\#)^2 + \#.
\]

Think of the above as an *equation with \(a\) as the unknown*, ans solve for \(a\).

**Step 3.** Replace \(a\) in the “preliminary” equation.
EXAMPLE. Find the quadratic function whose graph has vertex \((-1, 2)\) and passes through the point \((1, 10)\).

SOLUTION: Here the vertex gives \(h = -1\) and \(k = 2\).

**Step 1.** The preliminary equation is

\[
y = a(x - (-1))^2 + 2,
\]

which is the same as

\[
y = a(x + 1)^2 + 2.
\]

**Step 2.** We replace \(x = 1\) and \(y = 10\), and we get

\[
10 = a(1 + 1)^2 + 2,
\]

which leads to the equation

\[
10 = 4a + 2.
\]

We obviously get \(a = 2\).

**Step 3.** The function is then given by

\[
f(x) = 2(x + 1)^2 + 2
\]
APPLIED PROBLEMS.

The meaning of the vertex, as the maximum or minimum point for the quadratic function, is often used to solve optimization problems.

**EXAMPLE.** The daily cost \( C \) of producing lamps at the ABC COmpany is given by

\[
C = 900 - 20x + .2x^2,
\]

where \( x \) is the number of units produced. How many lamps should be produced in order to yield the minimum possible cost?

**Solution:** What we are dealing with here is a quadratic function

\[
f(x) = 0.2x^2 - 20x + 900,
\]

whose coefficients are \( a = 0.2(> 0) \), \( b = -20 \) and \( c = 900 \). What we need to find is the value of \( x \), for which \( f(x) \) takes the minimum value. Since \( a > 0 \), we know that \( f(x) \) has a minimum point at the vertex. So what we need to find is precisely the \emph{x-coordinate of the vertex}, that is the \emph{h-number}. So the answer is

\[
x = h = -\frac{b}{2a} = -\frac{-20}{2(0.2)} = \frac{20}{0.5} = 50.
\]