INVERSE FUNCTIONS

DEFINITION. Given two functions $f$ and $g$, we say that $g$ is the inverse of $f$, if one has:

(a) $f(g(x)) = x$, for all $x$ in the domain of $g$;
(b) $g(f(x)) = x$, for all $x$ in the domain of $f$.

In this case the function $g$ is simply denoted by $f^{-1}$.

Note. It is required that

• $f(x)$ takes values in the domain of $g$;
• $g(x)$ takes values in the domain of $f$.

KEY FEATURE: $f^{-1}(x) = ?$ means $f(?) = x$

EXAMPLE 1 (Informal). $f(x) = \frac{x + 2}{3}$.

To find the inverse function $f^{-1}$, we “undo” the operations:

$$g(x) = 3x - 2.$$ 

We check that the function $g$ is indeed the inverse of $f$.

(a) $f(g(x)) = \frac{g(x) + 2}{3} = \frac{3x - 2 + 2}{3} = \frac{3x}{3} = x$;

(b) $g(f(x)) = 3[f(x)] - 2 = 3 \left[ \frac{x - 2}{3} \right] + 2 = \frac{3(x - 2)}{3} + 2 = (x - 2) + 2 = x$.

Only after we checked these two conditions, it is safe to write:

$$f^{-1}(x) = 3x - 2.$$
**ALGEBRAIC METHOD.** To find the inverse of a function $f(x)$ algebraically, we proceed as follows:

**Step 1:** Set up the equation

$$f(y) = x,$$  \hspace{1cm} (1)

and solve for $y$. The solution of this equation should be of the form

$$y = \text{expression in } x.$$  \hspace{1cm} (2)

**WARNING!** The equation (1) may have several solutions. Keep only those $y$’s that are in the domain of $f$.

**Step 2:** Analyze results in Step 1. There are two cases:

(i) If there is some $x$ for which Step 1 produced two (or more) $y$’s, then STOP. The function $f$ does not have an inverse.

(ii) Define the “candidate” function $g(x)$ as the right hand side of (2). The domain of $g$ consists of all $x$’s for which Step 1 produces exactly one value of $y$.

**Step 3.** Check that $g$ is the inverse of $f$, using the conditions (a) and (b) in the definition.
EXAMPLE 2. Consider the function \( f(x) = \sqrt{x - 2}, \) and find its inverse (if it has one).

SOLUTION. First we need to determine the domain of \( f. \) This is

\[ \text{all numbers } \geq 2. \]

**Step 1.** Set up the equation \( f(y) = x, \) which reads:

\[ \sqrt{y - 2} = x, \]

and solve for \( y. \) Note that this equation forces \( x \geq 0. \) We get

\[ y = x^2 + 2. \]

We must restrict to the case when \( y \) is in the domain of \( f, \) that is, \( y \geq 2. \) This means \( x^2 + 2 \geq 2, \) which works without any further restrictions on \( x. \)

**Step 2.** The “candidate” function is:

\[ g(x) = x^2 + 2, \quad x \geq 0. \]

**Step 3.** (a) \( f(g(x)) = \sqrt{g(x)} - 2 = \sqrt{x^2 + 2} - 2 = \sqrt{x^2} = |x| = x, \) for all \( x \geq 0. \) (IT KEY THAT \( x \geq 0. \))

(b) \( g(f(x)) = [f(x)]^2 + 2 = [\sqrt{x - 2}]^2 + 2 = [x - 2] + 2 = x, \)

for all \( x \geq 2. \)

**Conclusion:** The function

\[ f(x) = \sqrt{x - 2}, \quad x \geq 2 \]

has an inverse, and its inverse is:

\[ f^{-1}(x) = x^2 + 2, \quad x \geq 0. \]
EXAMPLE 3. Consider the function $f(x) = x^2 - 1$, and find its inverse (it it has one).

SOLUTION. The domain of $f$ consists of all numbers.

Step 1. Set up the equation $f(y) = x$, which reads:

$$y^2 - 1 = x,$$

and solve for $y$. We have $y^2 = x + 1$, so we get

$$y = \pm \sqrt{x + 1}.$$

This forces $x \geq -1$. Note that both $y$’s are in the domain of $f$.

Step 2. We see that if we take for instance $x = 3$, then Step 1 produces two values $y = \pm 2$, so at this point we conclude that $f$ does not have an inverse.
EXAMPLE 4. Consider the function
\[ f(x) = x^2 - 1, \quad x \geq 0, \]
and find its inverse (if it has one).

SOLUTION. This function is the same as in Example 3, except that the \textit{domain of }f\textit{ consists of all numbers }\geq 0.\textit{ }

\textbf{Step 1.} Set up the equation \( f(y) = x \), which reads:
\[ y^2 - 1 = x, \]
and solve for \( y \). We have \( y^2 = x + 1 \), so we get
\[ y = \pm \sqrt{x + 1}. \]
This forces \( x \geq -1 \). We only need to keep those \( y \)'s that are in the \textit{domain of }f\textit{, that is, }y \geq 0\textit{. This means that we must have}
\[ y = \sqrt{x + 1}. \]

\textbf{Step 2.} The “candidate” function is:
\[ g(x) = \sqrt{x + 1}, \quad x \geq -1. \]

\textbf{Step 3. (a)} \( f(g(x)) = [g(x)]^2 - 1 = [\sqrt{x + 1}]^2 - 1 = [x + 1] - 1 = x \), for all \( x \geq -1 \).

\( \textbf{(b)} \) \( g(f(x)) = \sqrt{|f(x)|} + 1 = \sqrt{|x^2 - 1|} + 1 = \sqrt{x^2} = |x| = x \), for all \( x \geq 0 \). (IT KEY THAT \( x \geq 0 \).)

\textbf{Conclusion:} The function
\[ f(x) = x^2 - 1, \quad x \geq 0 \]
has an inverse, and its inverse is:
\[ f^{-1}(x) = \sqrt{x + 1}, \quad x \geq -1. \]
GRAPHING TECHNIQUES

It is possible to decide when a function has an inverse, by inspecting its graph.

**HORIZONTAL LINE TEST:** A function $f(x)$ has an inverse, precisely when there is no horizontal line that intersects the graph of $f(x)$ more than once.

In addition to this, if $f(x)$ has an inverse, then the domain of $f^{-1}$ consists of those numbers $\#$, for which the horizontal line $y = \#$ intersects the graph of $f(x)$.

Let us analyze Examples 2, 3, and 4 above.

**EXAMPLE 2.** $f(x) = \sqrt{x - 2}, \quad x \geq 2$. The graph is:

![Graph of f(x) with horizontal line y = # intersecting the graph]

We see that there is no horizontal line that intersects the graph twice (or more), so $f(x)$ indeed has an inverse. The horizontal lines that intersect the graph of $f(x)$ correspond to numbers $\geq 0$, so we can conclude that the domain of $f^{-1}$ consists of all numbers $\geq 0$. 

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EXAMPLE 3. \( f(x) = x^2 - 1, \quad x = \text{any number} \). The graph is:

![Graph of a function with a horizontal line at y = 3 intersecting the graph twice, indicating the function does not have an inverse.]

We see that the horizontal line \( y = 3 \) intersects the graph twice, so the function \( f(x) \) does not have an inverse.
**EXAMPLE 4.** $f(x) = x^2 - 1, \quad x \geq 0$. The graph is:

We see that there is no horizontal line that intersects the graph twice (or more), so $f(x)$ indeed has an inverse. The horizontal lines that intersect the graph of $f(x)$ correspond to numbers $\geq -1$, so we can conclude that the domain of $f^{-1}$ consists of all numbers $\geq -1$. 
**GRAPH OF THE INVERSE FUNCTION**

**REFLECTION RULE:** If the function $f(x)$ has an inverse, then the graph of the inverse function $f^{-1}(x)$ is obtained by reflecting the graph of $f(x)$ with respect to the diagonal line $y = x$.

Let us analyze Examples 2 and 4 above. (The diagonal line $y = x$ is shown as a black dotted line.)

**EXAMPLE 2.** $f(x) = \sqrt{x - 2}, \ x \geq 2$. The graph of $f(x)$ is shown in blue. The graph of the inverse function

$$f^{-1}(x) = x^2 + 2, \ x \geq 0$$

is shown in red.
EXAMPLE 4. $f(x) = x^2 - 1, \quad x \geq 0$. The graph of $f(x)$ is shown in blue. The graph of the inverse function

$$f^{-1}(x) = \sqrt{x + 1}, \quad x \geq -1$$

is shown in red.