RATIONAL EQUATIONS

Rational Equations are equations of the form:

\[
\text{Rational Expression in } x = \text{Rational Expression in } x. \quad (\ast)
\]

**Tips:** When solving a rational equation, use the following procedure.

- **Transform** (by subtracting all terms from one side from the other) the equation into:

  \[
  \text{Rational Expression in } x = 0.
  \]

- **Simplify the left-hand side,** so that the equation reads:

  \[
  \text{Fraction} = 0.
  \]

  At this point the equation will reduce to a polynomial equation:

  \[
  \text{Numerator} = 0. \quad (1)
  \]

- **Solve the polynomial equation** (1).

- **Validate all solutions of** (1) **into the original equation** (\ast). **Be aware that some solutions might have to be dismissed.** (This happens when some of the denominators in the original equation are zero.)

**EXAMPLE 1:** Consider the equation

\[
\frac{3x}{x^2 - 1} + \frac{x}{x^2 - 2x + 1} = 4. \tag{2}
\]

We start off by subtracting 4 from both sides. Since we ultimately want to combine the three fractions, we are also going to factor all denominators

\[
\frac{3x}{(x + 1)(x - 1)} + \frac{x}{(x - 1)^2} - 4 = 0. \tag{3}
\]

The common denominator is \((x + 1)(x - 1)^2\), so in order to convert all terms in (3) to fractions all having this denominator, we multiply their numerators and denominators accordingly:

\[
\frac{3x(x - 1)}{(x + 1)(x - 1)} + \frac{x(x + 1)}{(x - 1)^2} - \frac{4(x + 1)(x - 1)^2}{(x + 1)(x - 1)^2} = 0. \tag{4}
\]

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\(^1\) In the lecture the right-hand side was mistakenly written as “3” instead of “4.”
The numerator of the first fraction is $3x(x - 1) = 3x^2 - 3x$. The numerator of the second fraction is $x(x + 1) = x^2 + x$. Finally, the numerator of the third fraction is

$$4(x + 1)(x - 1)^2 = 4(x + 1)(x^2 - 2x + 1) = 4(x^3 - 2x^2 + x + x^2 - 2x + 1) = 4(x^3 - x^2 - x + 1) = 4x^3 - 4x^2 - 4x + 4.$$  

With these calculations in mind, the equation (4) now reads:

$$\frac{3x^2 - 3x}{(x + 1)(x - 1)} + \frac{x^2 + x}{(x - 1)^2} - \frac{4x^3 - 4x^2 - 4x + 4}{(x + 1)(x - 1)^2} = 0,$$

so after combining the numerators we get the equation (notice the use of grouping symbols):

$$\frac{[3x^2 - 3x] + [x^2 + x] - [4x^3 - 4x^2 - 4x + 4]}{(x + 1)(x - 1)^2} = 0.$$  

Opening up (carefully) the groups yields

$$\frac{3x^2 - 3x + x^2 + x - 4x^3 + 4x^2 + 4x - 4}{(x + 1)(x - 1)^2} = 0,$$

and finally combining the like terms we get

$$\frac{-4x^3 + 8x^2 + 2x - 4}{(x + 1)(x - 1)^2} = 0.$$  

Up to this point we have only carried on the first step (!), that is, we converted the equation to the form “Fraction = 0,” so our equation will now reduce to the polynomial equation “Numerator = 0,” which reads:

$$-4x^3 + 8x^2 + 2x - 4 = 0.$$  

(5)

Since this equation is not of a known type (linear, quadratic, or power), we should consider factoring the left-hand side. It turns out that the grouping method does work:

$$-4x^3 + 8x^2 + 2x - 4 = -4x^2(x - 2) + 2(x - 2) = 2(x - 2)(-2x^2 + 1).$$  

Since the equation (5) can be presented in factored form as:

$$2(x - 2)(-2x^2 + 1) = 0,$$
we can solve it by setting “Each Factor = 0.” (The constant factor 2 cannot be zero, so only two equations will be considered.) Setting $x - 2 = 0$ yields $x_1 = 2$. Setting $-2x^2 + 1 = 0$ leads to the power equation $x^2 = \frac{1}{2}$, which has as solutions $x_2 = \frac{1}{\sqrt{2}}$ and $x_3 = -\frac{1}{\sqrt{2}}$.

Since none of these values poses any problem in the original equation (2), we reach the following

**Conclusion:** The equation (2) has three real solutions $x_1 = 2$, $x_2 = \frac{1}{\sqrt{2}}$, and $x_3 = -\frac{1}{\sqrt{2}}$.

EXAMPLE 2: Consider the equation

$$8x^2 - 2x + 4 = x + 2 + 24x^3 + 8.$$  

We start off by subtracting the terms from the right-hand side. As in Example 1, we want all denominators factored. Using the **Sum of Cubes Formula**


the denominator in the second fraction in the right-hand side factors as: $x^3 + 8 = (x + 2)(x^2 - 2x + 4)$. With this factorization in mind, the equation (6) becomes

$$
\frac{8}{x^2 - 2x + 4} - \frac{x}{x + 2} - \frac{24}{(x + 2)(x^2 - 2x + 4)} = 0.  
$$

The common denominator is $(x + 2)(x^2 - 2x + 4)$, so in order to convert all terms in (7) to fractions all having this denominator, we multiply their numerators and denominators accordingly:

$$
\frac{8(x + 2)}{(x + 2)(x^2 - 2x + 4)} - \frac{x(x^2 - 2x + 4)}{(x + 2)(x^2 - 2x + 4)} - \frac{24}{(x + 2)(x^2 - 2x + 4)} = 0.  
$$

The numerator of the first fraction is $8(x + 2) = 8x + 16$. The numerator of the second fraction is $x(x^2 - 2x + 4) = x^3 - 2x^2 + 4x$. With these calculations in mind, the equation (8) now reads:

$$
\frac{8x + 16}{(x + 2)(x^2 - 2x + 4)} - \frac{x^3 - 2x^2 + 4x}{(x + 2)(x^2 - 2x + 4)} - \frac{24}{(x + 2)(x^2 - 2x + 4)} = 0.  
$$

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2 Since we expect several solutions, we use subscripts, such as $x_1$, $x_2$, and so on.

3 The **Sum of Cubes Formula** is: $A^3 + B^3 = (A + B)(A^2 - AB + B^2)$. This is used here with $A = x$ and $B = 2$. 

3
so after combining the numerators we get the equation (notice the use of grouping symbols):

\[
\frac{[8x + 16] - [x^3 - 2x^2 + 4x] - [24]}{(x + 2)(x^2 - 2x + 4)} = 0.
\]

Opening up (carefully) the groups yields

\[
\frac{8x + 16 - x^3 + 2x^2 - 4x - 24}{(x + 2)(x^2 - 2x + 4)} = 0.
\]

and finally combining the like terms we get

\[
\frac{-x^3 + 2x^2 + 4x - 8}{(x + 2)(x^2 - 2x + 4)} = 0. \quad (9)
\]

As explained in Example 1, the above equation is just the “Fraction = 0” equation. We now set up the polynomial equation “Numerator = 0,” which reads:

\[
-x^3 + 2x^2 + 4x - 8 = 0. \quad (10)
\]

As in Example 1, we factor the left-hand side using the grouping method:

\[
-x^3 + 2x^2 + 4x - 8 = -x^2(x - 2) + 4(x - 2) = (4 - x^2)(x - 2).
\]

Since the equation (10) can be presented in factored form as:

\[
(4 - x^2)(x - 2) = 0,
\]

we can solve it by setting “Each Factor = 0.” Setting \(4 - x^2 = 0\) leads to the power equation \(x^2 = 4\), which has as solutions \(x_1 = 2\) and \(x_2 = -2\). Setting \(x - 2 = 0\) yields again 2 as a solution, which has already appeared.

Up to this point, we have two candidates \((x_1 = 2\) and \(x_2 = -2\)) for solutions. As it turn out, the solution \(x_2 = -2\) is not a valid one, since it does not check the original equation (6). (The first fraction in the right hand side does not permit \(x = -2\).)

**Conclusion:** The equation (6) has only one real solution: \(x_1 = 2\).

**COMMENT:** The numerator from equation (9) can in fact be completely factored, using the **Difference of Squares Formula**

\[
-x^3 + 2x^2 + 4x - 8 = (4 - x^2)(x - 2) = (2 + x)(2 - x)(x - 2) = -(x+2)(x-2)^2.
\]
This means that the “Fraction = 0” equation (9) has in fact the form
\[
\frac{-(x + 2)(x - 2)^2}{(x + 2)(x^2 - 2x + 4)} = 0,
\]
but now the left-hand side can be simplified, thus giving the equation:
\[
\frac{-(x - 2)^2}{x^2 - 2x + 4} = 0.
\]
This way, if we set the equation “Numerator = 0,” we would only get
\[
-(x - 2)^2 = 0,
\]
which has only one solution \(x = 2\). This discussion brings up the importance of simplifying rational expressions, especially after we reduced the equation to the form “Fraction = 0.”