1. This problem is very hard to typeset, but the answers are:

(a) $Q(x) = x^2 + 3x + 9$ and $R(x) = 7$. To check, we compute

\[(x - 3)Q(x) + R(x) = (x - 3)(x^2 + 3x + 9) + 7 = x^3 - 3x^2 + 3x^2 - 9x + 9x - 27 + 7 = x^3 - 20.\]

(b) $Q(x) = 2x^3 + 6x^2 - x - 9$ and $R(x) = -20x + 11$. Again, to check, we compute

\[(x^2 - 3x + 1)Q(x) + R(x) = (x^2 - 3x + 1)(2x^3 + 6x^2 - x - 9) + (-20x + 11) = 2x^5 - 6x^4 + 2x^3 + 6x^4 - 18x^3 + 6x^2 + x^3 - 3x^2 - x - 9x^2 + 27x - 9 - 20x + 11 = 2x^5 + 0x^4 - 17x^3 + 0x^2 + 6x + 2 = 2x^5 - 17x^3 + 6x + 2.\]

2. Let us construct a polynomial $P(x)$, having the desired properties. First, we note that any polynomial with a factor of $(x + 1)^2$, will have a zero of multiplicity 2 at $-1$. So, $P$ should have a factor of $(x + 1)^2$. Now, if we want there to be a root at $2 + \sqrt{5}$, we should also have a factor of $(x - (2 + \sqrt{5}))$. In order that $P(x)$ have rational coefficients, we must have $2 - \sqrt{5}$ being a root of $P(x)$, as well, so that $(x - (2 - \sqrt{5}))$ is a factor of $P(x)$. (See the boxed statements on page 317 for a precise formulation of this result.) Since $P(x)$ is to have degree 4 and we have just determined 4 linear
factors, it must be that

\[ P(x) = (x + 1)^2(x - (2 + \sqrt{5}))(x - (2 - \sqrt{5})). \]

3. \( f(x) = 2x^4 - x^3 - 13x^2 + 5x + 15. \)

(a) Since we are looking for rational zeros, it is reasonable to try to implement the rational zero theorem. This theorem states that if \( p/q \) is a zero of \( f(x) \), where \( p \) is relatively prime to \( q \) (i.e. have no common factors except \(-1 \) and \(1\)) then \( p \) is a factor of the constant coefficient of \( f \) and \( q \) is a factor of the leading coefficient of \( f \). So, we find the factors of the leading coefficient 2 and the constant coefficient 15. 2 has factors \( \pm 1, \pm 2 \) and 15 has factors \( \pm 1, \pm 3, \pm 5, \pm 15 \). It follows that the possible rational zeros of \( f \) are:

\[
\frac{\pm 1, \pm 3, \pm 5, \pm 15}{\pm 1, \pm 2} = \pm 1, \pm 3, \pm 5, \pm 15, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \pm \frac{15}{2}.
\]

Among these possibilities, direct substitution shows that \(-1 \) and \( \frac{3}{2} \) are the only rational zeros.

(b) Since \(-1 \) and \( \frac{3}{2} \) are zeros of \( f \), we know that \((x + 1) \) and \((x - \frac{3}{2}) \) are factors of \( f \). That is, \( f(x) = (x + 1)(x - \frac{3}{2})Q(x) \), for some polynomial \( Q(x) \). If we can find the zeros of \( Q(x) \), we will know all of the zeros of \( f(x) \). So, to find \( Q(x) \), we use long division to calculate \( f(x)/[(x + 1)(x - \frac{3}{2})] = f(x)/(x^2 - \frac{1}{2}x - \frac{3}{2}) \). This gives \( Q(x) = 2x^2 - 10 = 2(x^2 - 5) \). So, the remaining zeros of \( f(x) \) are when \( Q(x) = 0 \), or equivalently, when \( x = \pm \sqrt{5} \).

4. (i) \( f(x) = x^4 + 5x^3 + 6x^2 = x^2(x^2 + 5x + 6) = x^2(x + 2)(x + 3) \)

(ii) | Zero | Multiplicity |
<table>
<thead>
<tr>
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<tbody>
<tr>
<td>0</td>
<td>2</td>
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<tr>
<td>-2</td>
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<td>-3</td>
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5. \( f(x) = x^4 + 3x^3 - x - 3. \)
(a) By the rational zero test, we find that \( f \) has a zero at \(-3\) and \(1\).

(b) Since \( f \) has zeros at \(-3\) and \(1\), we see that \((x + 3)(x - 1) = x^2 + 2x - 3\) must divide \( f \). Upon long division, we find that

\[
f(x) = (x^2 + 2x - 3)(x^2 + x + 1).\]

(c) We already have two linear factors for \( f \), namely, \((x + 3)\) and \((x - 1)\). To find the others, we must find the roots of \(x^2 + x + 1\). Unfortunately, we this does not have real roots, but the quadratic equation shows that the roots of \(x^2 + x + 1\) are

\[
x = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}i}{2}.
\]

So, \( f \) factors into the following \( f(x) = (x + 3)(x - 1)\left(x - \frac{-1 + \sqrt{3}i}{2}\right)\left(x - \frac{-1 - \sqrt{3}i}{2}\right)\).

6. (a) Remember inequality signs change only when you multiply or divide by a negative.

\[
-3 \leq 5 - 2x < 7 \iff -8 \leq -2x < 2 \quad \text{(by subtracting 5)}
\]

\[
\iff 4 \geq x > -1 \quad \text{(by dividing by -2)}
\]

In interval notation, this is the set \((-1, 4]\).

(b)

\[
5(2x - 1) + 2 > 3(x + 9) - 2x \iff 10x - 5 + 2 > 3x + 27 - 2x
\]

\[
\iff 10x - 3 > x + 27
\]

\[
\iff 9x > 30
\]

\[
\iff x > \frac{10}{3}.
\]

In interval notation, this is the set \((\frac{10}{3}, \infty)\).
7. \[
|2x + 5| \leq 2 \iff 2x + 5 \leq 2 \text{ and } 2x + 5 \geq -2 \\
\iff -2 \leq 2x + 5 \leq 2 \\
\iff -7 \leq 2x \leq -3 \\
\iff \frac{-7}{2} \leq x \leq -\frac{3}{2}.
\]

8. In general, a polynomial of degree \( n \) can have at most \( n \) \( x \)-intercepts (i.e. zeros). If we know in addition whether \( n \) is even or odd, we can get more information. Namely, if \( n \) is even, then the number of \( x \)-intercepts can be anywhere between 0 and \( n \). On the other hand, if \( n \) is odd, then the number of \( x \)-intercepts will be between 1 and \( n \). The number of “turning points” can be anywhere between 0 and \( n - 1 \). If \( n \) is odd, then \( \lim_{x \to \infty} f(x) = -\lim_{x \to -\infty} f(x) \). If \( n \) is even, then \( \lim_{x \to \infty} f(x) = \lim_{x \to -\infty} f(x) \).

In our particular problem, the above statements imply that \( f \) can have anywhere between 1 and 23. \( f \) can have anywhere between 0 and 22 turning points. Since 23 is odd, \( f(x) \to \infty \) as \( x \to -\infty \), we must have \( f(x) \to -\infty \) as \( x \to \infty \).

9. Let us write 
\[
f(x) = \frac{x^3 - 4x}{x^2 - 1} = \frac{x(x-2)(x+2)}{(x-1)(x+1)}.
\]
(a) \( f \) has zeros only when the numerator is zero \textit{and} the denominator is nonzero. These values are when \( x = 0, -2, 2 \). The \( y \)-intercept occurs when \( x = 0 \). In point notation, the intercepts of \( f \) are \((0,0), (-2,0), \) and \((2,0)\).

(b) Vertical asymptotes will occur for values of \( x \) when the denominator is equal to zero and the numerator is not. These are the lines \( x = -1 \) and \( x = 1 \).

Since the degree of the polynomial in the numerator is one greater than the degree of the polynomial in the denominator, we must have \( f \) having an oblique asymptote. That is, \( f \) is asymptotic to some line \( y = mx + b \). (Note that this immediately tells us that there are no horizontal asymptotes.) To find this line, we use long division
to find $Q(x)$ and $R(x)$ such that $x^3 - 4x = (x^2 - 1)Q(x) + R(x)$. This process gives $Q(x) = x$ and $R(x) = -3x$. The line $y$ that $f$ is asymptotic to is given by $Q(x)$. That is, $f$ is asymptotic to the line $y = x$.

(c) $f(-x) = \frac{(-x)^3 - 4(-x)}{(-x)^2 - 1} = \frac{-x^3 + 4x}{x^2 - 1} = -\frac{x^3 - 4x}{x^2 - 1} = -f(x)$. So, $f$ is symmetric about the origin.

(d) $f$ is positive in the intervals $(-2, -1)$, $(0, 1)$ and $(2, \infty)$.

10. (a) Following the procedure outlined in 9b, we have that $f$ has an oblique asymptote at the line $y = x - 1$ and a vertical asymptote at $x = -2$. Since $f$ has an oblique asymptote $f$ can not have any horizontal asymptotes.

(b) $g$ has a horizontal asymptote when $y = 0$, since the degree of the numerator is strictly less than the degree of the denominator. (In the even that the two degrees are equal, there will still be a horizontal asymptote and it will be at the line $y = \frac{a}{b}$, where $a$ is the leading coefficient of the polynomial in the numerator and $b$ is the leading coefficient of the denominator.) There are two vertical asymptotes at $x = \sqrt{3}$ and $x = -\sqrt{3}$.

11. (a) $f(x) \to \infty$ as $x \to -\infty$, since the degree of $f$ is even and the leading coefficient is positive.

(b) $g(x) \to -\infty$, since the degree of $g$ is odd and the leading coefficient is positive.