The idea is that one has output \( y = f(x) \) from function (machine) \( f \) and needs to recover the input \( x \). That is, we need to solve the equation

\[
(1) \quad y = f(x)
\]

for \( x \) in terms of \( y \). This may not be possible. For example, the machine \( f(x) = x^2 \) outputs 16 from numerous inputs (notably from \( x = 4 \) and from \( x = -4 \)), so from \( y = 16 \) we can't work back to the \( x \) for which \( f(x) = 16 \), we can't solve eq. (1) for \( x \) in terms of \( y \). Success in doing so amounts to expressing \( x \) as a formula involving \( y \), that is, as a function \( g(y) \), so

\[
(2) \quad x = g(y).
\]

The special function \( g \) that does this for \( f \) gets a special name, the inverse (think of reverse) of \( f \), and a special symbol \( f^{-1} \). Thus (2) is usually written

\[
(1)' \quad x = f^{-1}(y).
\]

Eqs (1) and (1)' really express the same relation. They say the same thing, rather like the sentences...
If a point \((x, y) = (x, f(x))\) lies on graph \(f\), then \((1)\)' says that the point \((y, x) = (y, f^{-1}(y))\) lies on graph \(f^{-1}\)

**Example 1**

\[
y = f(x) = x^3 + 1.
\]

For this \(f\) we can recover \(x\) from \(y\):

\[
x^3 = y - 1 \quad x = \sqrt[3]{y - 1}
\]

that is,

\[
(1)' \quad x = f^{-1}(y) = \sqrt[3]{y - 1}
\]

points \((-1, 0) = (-1, f(-1))\) points \((0, 1) = (0, f(0))\) lie on graph \(f\)

\((2, 9) = (2, f(2))\)

\((0, -1) = (0, f^{-1}(1))\)

points \((1, 0) = (1, f^{-1}(1))\) lie on graph \(f^{-1}\)

\((9, 2) = (9, f^{-1}(9))\)

**NOTE:** The custom of labelling input with \(x\) and output with \(y\) is so entrenched that we are uncomfortable with a recipe like \((1)'\) in which these roles are reversed, so we like to re-write
that recipe (formula 2) is

\[ f^{-1}(x) = \sqrt[3]{x-1}. \]

Of course it's the same recipe: In words (1)', (1)" both say “subtract 1 from the input and take the cube root of the result” to get the output of \( f^{-1} \) for that input.

Remember too (from §4.1) that the graph of \( f^{-1} \) is that of \( f \) rotated 180° about the diagonal line \( y=x \). See p. 44.

This extensive review is preparation for the real goal of this section, which is to study the inverse of the exponential function. So fix a base \( b > 1 \) and look at the exponential function

\[ y = E_b(x) = b^x \]

and solve this equation for \( x \) in terms of \( y \), that is, undo the exponential function.

The graph shows this is possible, that is, \( y \) determines \( x \).
The graph of $f^{-1}$ is that of $f$ rotated out of this plane through $180^\circ$ around the diagonal line. This move changes every $(a, b)$ point on graph $f$ into a $(b, a)$ point on graph $f^{-1}$.

$$f(f^{-1}(x)) = x = f^{-1}(f(x))$$

Each function undoes the other and returns the original input.
The "undoing" function that expresses $x$ in terms of $y$ is called the logarithmic function to base $b$ and we write it

$$x = L_b(y) = \log_b(y), \quad L_b = E_b^{-1}$$

The fact that this function undoes the exponential function is expressed by the statement

$$y = b^x \text{ means the same thing as } x = \log_b(y).$$

It can also be expressed by two equations:

$$\log_b(b^x) = x \quad \text{for all } x, \quad b^{\log_b y} = y \quad \text{for all positive } y.$$ 

Do you see why only positive $y$ are allowed? It's because the graph of $y = b^x$ involves only positive $y$, the exponential function only outputs positive numbers.

These facts are summarized in the green boxes on p. 289, 291 of the text.

Now the graph of this new logarithm function is related to that of the exponential fct. just as the graphs of $f$ and $g$, above. Namely, we reflect the graph of $y = b^x$ in the diagonal line $y = x$ to get the graph of $x = \log_b y$.

The box at the bottom of p. 290 of the text summarizes the graph and properties of the logarithmic function very efficiently:
\[ y = b^x \]

\[ d = b^c \]

\[ (c, b^c) = (\log_b d, d) \]

\[ (b, 1) \]

\[ c = \log_b d \]

\[ (b, c) = (d, \log_b d) \]

\[ (0, 1) \]

\[ x = \log_b (y) \]
For $b > 1$ and $y = \log_b(x)$

- **Domain**: Eligible $x = \text{all positive nos.} = (0, \infty)$
- **Range**: $y$ output = all nos. $= (-\infty, \infty)$
- **X-intercept**: $(1, 0)$; there is no y-intercept
- **Graph**: Increases (rises) as $x$ increases
- **$y$ is negative when** $0 < x < 1$
- **$y$ is positive when** $1 < x$
- **$y$-axis is a vertical asymptote**: more precisely $y = \log_b(x) \to -\infty$ as $x \to 0^+$ ($x$ approaches 0 from the right)
- **No horizontal asymptote**, but $y = \log_b(x) \to \infty$ as $x \to \infty$
- **Graph** is the reflection of graph of $y = b^x$ in the diagonal line.
The three bases most used are

\( b = 10 \), because in our decimal system powers of 10 play a special role; \( \log_{10} \) is called the common logarithmic function.

\( b = 2 \), because computers use binary numbers.

\( b = e \), because (as we've seen) the exponential to base \( e \) comes up everywhere in describing growth and decay phenomena in nature—as well as continuously compounded interest—also a growth phenomenon. \( \log_e \) is called the natural logarithmic function and it's often written \( \ln \).

**Example 2.** Find \( \log_2 \left( \frac{1}{8} \right) \).

**Sol.** \( y = \log_2 \left( \frac{1}{8} \right) \) means by (4) that \( 2^y = \frac{1}{8} \). But \( \frac{1}{8} = \frac{1}{2^3} = 2^{-3} \); so the number \( y \) that does this job is \( y = -3 \). We can also look at this from the point of view of (5):

\[
\log_2 \left( \frac{1}{8} \right) = \log_2 \left( 2^{-3} \right) = -3.
\]

**Example 3.** If \( x > 0 \), express \( \log_b(x^2) \) in terms of \( \log_b(x) \).

**Sol.** \( \log_b(x) = y \) means by (6) that \( b^y = x \), and so \( b^{2y} = (b^y)^2 = x^2 \). By (4) this eq. means that \( 2y = \log_b(x^2) \). Thus we see (cf. power rule p. 302)

\[
\log_b(x^2) = 2y = 2 \log_b(x).
\]
Note on Notation:

\[ \log_e, \text{ the natural logarithm,} \]

is denoted often by

1. LN

\[ \log_{10} \] is often denoted simply

2. LOG

The symbols 1. and 2. are what the TI-83 keyboard uses—in its left-hand column.
Example 4. What is the domain and graph of the function
\[ f(x) = \log_2 (1-x) \, ? \]

Solution: First let's analyze
\[ g(x) = \log_2 (-x). \]

The only eligible input for a logarithmic function is positive numbers. Hence for \( g \) we need 
\(-x > 0\), that is, \( x < 0 \). Recall from the p. 116 box that the graph of \( g \) is that of \( \log_2 \) reflected in the y-axis:

The graph of \( f \) is gotten by shifting the graph of \( g \) right 1 unit, because
\[ f(x) = \log_2 (1-x) = \log_2 (-(-x-1)) = g(x+1). \]
§4.3

\[ y = f(x) = \log_2(1-x) = g(x-1) \]

\((-1, t)\)

\[ 2^{-x} = 1 \]
EXAMPLE 5. Specify the domain and sketch the graph of \( f(x) = \log_{10}(x^2) \).

Sol. \( \log_{10}(t) \) makes sense just if \( t > 0 \). Since \( t = x^2 \) satisfies \( x^2 > 0 \) for all \( x \neq 0 \), the domain of \( f \) is \( \mathbb{R} \setminus \{0\} \). Notice \( f \) is an even function:

\[
f(-x) = f(x),
\]

so its graph is symmetric with respect to the y-axis, and we need only graph it for \( x > 0 \). For such \( x \),

\[
f(x) = \log_{10}(x^2) = 2 \log_{10}(x),
\]
as we saw in Example 3.
Change of Base (see box p. 294)

We have seen that the graphs of the exponential function $E_b$ are pretty much the same for all $b > 1$, as therefore the graphs of their inverses $L_b = \log_b$ must be. The similarity is dramatic and simple. If $a > 1$ and $b > 1$, then the graph of $L_a$ is that of $L_b$ stretched by a definite factor in the vertical direction, as (8) below will show. So, suppose we have the graph of $L_b$ (or a table of base $b$ logs) and we want the graph of $L_a$ (or a table of base $a$ logs): For each $x > 0$ we know $L_b(x)$ and we want $y = L_a(x)$. Remember (4) that

$$y = L_a(x) \text{ means } x = a^y$$

But by (5)

$$a = E_b(L_b(a)) = b^{L_b(a)}$$

so (6) reads

$$x = (b^{L_b(a)})^y = b^{yL_b(a)}$$

using the power rule (2nd box p. 9) of exponents.

According to (5) eq. (7) is telling us that

$$L_b(x) = yL_b(a)$$

and solving this for $y$ gives

$$L_a(x) = y = \frac{L_b(x)}{L_b(a)}$$
In words (8) says take the single number $L_b(a)$ and divide every $L_b(x)$ by it to get $L_a(x)$. From a table of $\log_b$ we automatically get a table of $\log_a$. There is no need to waste precious circuitry and memory equipping TI-83 with both LOG ($\log_{10}$) and LN ($\log_e$) keys. Why did those clever people in Dallas do so?

Conversion between $\log_e$ and $\log_{10}$ is the one most encountered. According to (8) this involves the factor

$$\log_e 10 \approx 2.30258509 \ldots$$

a number that pops up often in scientific literature.