§2.1 Solving Equations & Word Problems

We've noted that some equations, like

(1) \[ x^2 - 1 = (x-1)(x+1) \]

are identities, valid for all \( x \) (as we confirm in this example by multiplying out the right side). By contrast

(2) \[ 3x + 1 = 0 \]

is a conditional equation, valid for certain \( x \) and not for others. Of course (2) is valid for \( x = -\frac{1}{3} \) and only for that \( x \). It is an example of a linear equation.

(3) \[ mx + b = 0 \]

where \( m, b \) are concrete numbers. If \( m = 0 \) and \( b \neq 0 \) then there are no \( x \) that satisfy (3). While if \( m = 0 \) and \( b = 0 \) then all numbers \( x \) satisfy (3). We put aside this degenerate case and suppose \( m \neq 0 \). Then the number \( \frac{1}{m} \) exists (box p. 5) and multiplying (3) by it produces the new equation

(4) \[ \frac{1}{m}(mx + b) = \frac{1}{m} \cdot 0 = 0, \]

that is,

\[ x + \frac{1}{m}b = 0 \]

which is logically equivalent to (3), i.e., makes the same statement about \( x \) as (3). Finally, adding \(-\frac{1}{m}b\) to both sides of (4) produces another logically equivalent equation

\[ x + \frac{1}{m}b + -\frac{1}{m}b = 0 + -\frac{1}{m}b, \]

that is.\]
\[ x = -\frac{b}{m}, \quad b = -\frac{b}{m}. \]

So this \( x \) and only this one satisfies conditional equation (3).

All this is trivial but illustrates what we do in general to solve an equation.

\[ f(x) = 0 \]

that is, to find the \( x \) which satisfy it (render it valid). These are the places where the graph of the eq. \( y = f(x) \) cuts the horizontal axis, the \( x \)-intercepts of the graph. These are also known as the zeros of the function \( f \). Usually there are several and altogether they comprise the zero-set of function \( f \) or the solution set of equation (6).

The linear equation (3) is misleadingly simple. Usually we have to factor \( f \) and solve (6) by finding the \( x \) that make at least one of the factors equal 0. The basis for this is the simple zero-principle, (p.153 of text):

\[ ab = 0 \text{ just if either } a = 0 \text{ or } b = 0. \]

Indeed, if \( ab = 0 \) but \( a \neq 0 \), we multiply by \( \frac{1}{a} \).
and convert the equation \( ab = 0 \) into
\[
b = \frac{1}{a}, \quad ab = \frac{1}{a} \cdot 0 = 0
\]
that is, if \( a \neq 0 \), then \( b \) must be 0.

**Example 1** \( f(x) = x^2 - 3x + 2 \).
Since \( f(x) \) factors as
\[
f(x) = (x-1)(x-2)
\]
solving equation \((a)\) means solving
\[
(x-1)(x-2) = 0
\]
According to principle \((a)\) this product can be 0 in just two ways:
\[
\begin{align*}
either & \quad x - 1 = 0 \quad \text{or} \quad x - 2 = 0 \\
& \quad x = 1 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad x = 2.
\end{align*}
\]
The solution set of \((a)\) in this example is \( \{1, 2\} \).

**Example 2** \( f(x) = x^2 - 5 \).
Here \( f(x) \) factors as \((x - \sqrt{5})(x + \sqrt{5})\), so solving \((b)\) means solving
\[
(x - \sqrt{5})(x + \sqrt{5}) = 0
\]
which according to principle \((a)\) means
\[
\begin{align*}
either & \quad x - \sqrt{5} = 0 \quad \text{or} \quad x + \sqrt{5} = 0 \\
& \quad x = \sqrt{5} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad x = -\sqrt{5}.
\end{align*}
\]
The solution set of \((b)\) in this example is \( \{-\sqrt{5}, \sqrt{5}\} \).

**Example 3** \( f(x) = x^2 + 5 \).
Whatever (real) number \( x \) may be, \( x^2 \geq 0 \), so \( x^2 + 5 \geq 5 \). Therefore the equation \( 0 = f(x) = x^2 + 5 \)
has no solution. (The solution set is empty.)
§2.1 Modelling (AKA Word or Story Problems)

Here is my 4-step procedure. The book (pp. 139-140) offers a more elaborate 5-step that you may prefer.

(I) (Adam & Eve phase) Give (symbolic) names to the things you’re asked to determine. Choose letters that suggest what they stand for: \( t \) for time, \( p \) for profit, etc. Never use a symbol you’ve not defined. Never use same symbol for different jobs.

(II) Record as equations (sometimes as inequalities) in terms of these symbols all the relations and quantitative info. the problem gives.

(III) Be conscious of units (esp. be consistent ‘don’t mix feet and inches’). Incorporate units into step II — as physicists & chemists do. After appropriate cancellations, units on the two sides of \( \approx \) should be same. If not, this is a helpful indication that you’ve set up the problem wrong!

(IV) Try to solve the equations in II for the things (unknowns) sought in terms of the other data. In general this is do-able if the no. of equations equals the no. of unknowns. Too few eqs (underdetermination) means more than 1 solution; too many eqs (overdetermination) may turn out inconsistent, resulting in there being no solution.
Example 4 (like Ex. 6 p.140) The total profit for a company in Feb. was 20% higher than in Jan. and the combined profit was $157,000. Find the profit for each month.

Sol. Let \( J \) (dollars) denote Jan's profit. We're given

\[
\text{Feb's profit} = J + 20\% J = J + 0.2J = 1.2J
\]

and we're given that

\[
157,000 = \text{total profit} = \text{Jan's profit} + \text{Feb's profit} = J + (1.2)J = 2.2J
\]

\[
J = \frac{157,000}{2.2} \approx 71,363.64 \text{ (dollars)}.
\]

Then

\[
\text{Feb's profit} = (1.2)J = (1.2)157,000 = 188,400 \approx 65,236.36 \text{ (dollars)}.
\]

Example 5 Find two consecutive whole numbers whose product is 5 less than the square of the smaller.

Sol. If \( n \) denotes a whole number, its successor is \( n+1 \) and \( n \) is the smaller of the two. We seek all \( n \) that satisfy

\[
\begin{align*}
(n)(n+1) &= (\text{smaller})^2 - 5 = n^2 - 5 \\
n^2 + n &= n^2 - 5 \\
0 &= 6 \\
\end{align*}
\]

\[
\begin{array}{c}
m = -5 \\
m+1 = -4
\end{array}
\]

Example 6 A livestock water-trough is 12 feet long, 3 feet wide and 3 feet deep. If it contains 70 gallons

\[
\begin{align*}
\text{Volume of water} &= 70 \\
\text{Volume of water} &= \text{length} \times \text{width} \times \text{depth} \\
70 &= 12 \times 3 \times 3 \\
70 &= 108 \\
\end{align*}
\]
of water and 1 gallon of water occupies \( .1336 \) cubic feet, what is the depth of the water?

\[ \text{Sol: When } d \text{ (feet) is the depth, the amount of water is } \]
\[ 3.12 \cdot d \text{ cubic feet.} \]
\[ \frac{1 \text{ gallon}}{.1336 \text{ cubic feet}} = \frac{36d}{.1336} \text{ gallons. (Note cancellation of units.)} \]

We're asked what \( d \) supplies 70 gallons, so we solve
\[ 70 = \frac{36d}{1336} \]
\[ d = \frac{(1336)70}{36} \approx .26 \text{ (feet) [\approx 3 inches].} \]

Example 7: \$12,000 is to be invested in a conservative fund paying \( 4 \frac{1}{2} \%) \) and in a somewhat riskier one paying \( 5 \%) \) in order to earn \$560 per year to pay for a daughter's college textbooks. What is the smallest amount that can be invested in the riskier fund and still accomplish this?

\[ \text{Solution: Let } H \text{ (dollars) be invested at the higher rate. Then } 12,000 - H \text{ (dollars) is invested at the lower rate. We seek the } H \]
\[ \text{that satisfy} \]
\[ 550 = \text{total annual earnings} \\
= \text{earnings @ high rate} + \text{earnings @ low rate} \\
= 5\% \text{ of high investment} + 4.5\% \text{ of low investment} \\
= .05H + .045(2000-H) \\
= .05H + .045(2000) - .045H \\
= (.050 -.045)H + 540 \\
= (.005)H + 540 \\
H = \frac{580 - 540}{.005} = 8000 \text{ (dollars).} \]

**Example 8.** A certain plant has fixed costs of $10,000 per month and variable costs of $9.30 per unit manufactured. The plant manager has $85,000 available to cover monthly costs. How many units can she afford to manufacture?

*Fixed costs are those that occur regardless of the level of production, such as property tax, while variable costs, like that of the raw material, depend on the level of production.*
Let $u$ denote the number of units manufactured per month, $C$ (dollars) the cost of producing them. Then $C$ is made up of two components:

$$C = \text{fixed costs} + \text{variable costs}$$

$$= 10000 \text{ dollars} + \frac{9.3 \text{ dollars}}{\text{unit}} \cdot u \text{ units}$$

$$= 10000 + (9.3)u \text{ dollars}.$$ 

The question is how many $u$ can be produced if $85000$ is available as $C$?

$$85000 = C = 10000 + (9.3)u$$

$$75000 = (9.3)u$$

$$u = \frac{75000}{9.3} \approx 8064.516129.$$ 

Since presumably a unit (e.g., a golfball) cannot be fractionated, the relevant $u$ must be

$$u = 8064 \text{ (units)}.$$ 

Example 9 (on relative speeds). A plane flies to a city 1500 mi. distant. After travelling the same amount of time on its return trip it is still 300 mi. from home. Its airspeed is 600 mph. What is the wind's speed?
Solution: Let \( t \) (hrs.) be time for the first leg, 
\( w \) (mi/hr) the speed of the tailwind.

\[
\begin{align*}
\frac{t}{1500} & \quad \rightarrow \quad W \\
\frac{t}{300} & \quad \leftarrow \quad \frac{1500-300}{1500-300} \quad W
\end{align*}
\]

For 1st leg speed of plane relative to ground = \( 600+W \)
For 2nd \( \quad \frac{600-W}{t} \) = \( 600-W \)

Since

\[
\text{rate} = \frac{\text{distance}}{\text{time}}
\]

we have

(1) \( 600+W = \frac{1500}{t} \) \quad and \quad (2) \( 600-W = \frac{1500-300}{t} \)

Divide eq. (2) by eq. (1) to eliminate \( t \) (which the problem doesn't ask about and which we don't know):

\[
\frac{600-W}{600+W} = \frac{\frac{1500-300}{t}}{\frac{1500}{t}} = \frac{1200}{1500} = \frac{4}{5}
\]

\[
5(600-W) = 4(600+W)
\]

\[
5(600-4\cdot600) = 5W + 4W
\]

\[
600 = 9W
\]

\[
W = \frac{600}{9} = 66 \frac{2}{3} \quad \text{(mi/hr.)}
\]

**Example:** On first leg of a 317 mi trip you average 58 mph, but traffic density reduces this to 52 mph for the final leg. If whole trip cost 5 hrs, 45 min., how much time was spent driving at each rate?
Solution: \[ \begin{array}{ccc} \text{fast} & & \text{slow} \\ x & & 317-x \end{array} \]

Let \( x \) (mi) be driven at 58 mi/hr average. Then \( 317-x \) (mi) is 52

Since \[ \text{distance} = \text{(rate)} \times \text{(time)} \]

(1) time for \( x \) = \( \frac{x \text{ mi}}{58 \text{ mi/hr}} \) = \( \frac{x}{58} \text{ hr} \)

(2) time for \( 317-x \) = \( \frac{(317-x) \text{ mi}}{52 \text{ mi/hr}} \) = \( \frac{317-x}{52} \text{ hr} \)

The total time is the sum of (1) and (2) and is \( 5 \frac{3}{4} = 5.75 \) hours:

\[ 5 \frac{3}{4} = \frac{x}{58} + \frac{317-x}{52} = \frac{x}{58} + \frac{317}{52} - \frac{x}{52} \]

\[ 5 \frac{3}{4} - \frac{317}{52} = \frac{1}{58} x - \frac{1}{52} x = \left( \frac{1}{58} - \frac{1}{52} \right) x \]

\[ x = \frac{5 \frac{3}{4} - \frac{317}{52}}{\frac{1}{58} - \frac{1}{52}} \text{ miles.} \]

After a little grade-school arithmetic this compound fraction simplifies to

\[ x = 174 \text{ (miles), at faster speed (58 mph)} \]

\[ 317-x = 143 \text{ (miles), at slower speed (52 mph)} \]

Again notice how the units behave in (1) and (2), resulting in each case in the appropriate final unit (for time).
Can you ascertain the distance to the horizon if you know the radius of the earth?

Let \( r \) denote the earth's radius, \( h \) your height from feet to eyeballs, \( d \) the distance from your eyes to the horizon, all measured in the same unit, szy meters. These data are related as in the figure.

Pythagoras says

\[
(r+h)^2 = r^2 + d^2 \quad (1).
\]

Consequently,

\[
d = \sqrt{(r+h)^2 - r^2} \quad (2).
\]

One finds \( r \) in any encyclopedia (it's \( \approx 6,378,000 \)). For me \( h \approx 1.7 \).

Hence for me the horizon is

\[
d = \sqrt{(6,378,001.7)^2 - (6,378,000)^2} \approx 4656.74 \text{ meters away.}
\]

It is possible to directly measure \( d \) and then use (1) to compute \( r \), and the ancient Greeks did so!!

Note: All this appears on a music CD by the Berlin Band Knorkator!
Is the Internet a Reliable Information Source?

I recently found the following in cyberspace:

Every 18 months a man shaves an area equal to that of a (European) football field.

This is impressive. Is it credible?

A European football field is 100 meters by 64 meters, so its area is $(100)(64) = 6400$ (meters)$^2$. Hence from (x) we can figure a man's daily shave as follows:

\[
\frac{6400 \text{ (meters)}^2}{1 \text{ football field}} \cdot \frac{1 \text{ football field}}{18 \text{ months}} \cdot \frac{1 \text{ month}}{30 \text{ days}} = \frac{6400 \text{ (meters)}^2}{(18)(30) \text{ days}} = 11.85 \frac{\text{(meters)}^2}{\text{day}}.
\]

But the adult human body has only about 2 (meters)$^2$ of skin surface!

The healthy skepticism that even modest "numeracy" confers. Such critical facility (and attitude!) is one of the goals of a general education course like MATH 100.