Choose and work any 6 of the following 15 problems. Start each problem on a new sheet of paper. Do not turn in more than six problems. In the problems below, a space always means a topological space.

1. Let \( f : X \to Y \) be any function, let \( A \subseteq \mathcal{P}(X) \) and \( B \subseteq \mathcal{P}(Y) \). Prove or disprove each of the following:
   (a) \( f[\bigcap A] = \bigcap\{f[A]|A \in A\} \)
   (b) \( f^{-1}[\bigcap B] = \bigcap\{f^{-1}[B]|B \in B\} \)

2. (a) State the Axiom of Choice.
    (b) State the Well-Ordering Theorem.
    (c) Either use the Axiom of Choice to prove the Well-Ordering Theorem or use the Well-Ordering Theorem to prove the Axiom of Choice.

3. Let \( X \) be a well-ordered set in the order topology, and assume \( X \) has a maximal element. Prove \( X \) is a compact space.

4. Prove that every closed subset of a metrizable space is a countable intersection of open sets.

5. Let \( X \) be a connected and locally path connected space. Prove \( X \) is path connected.

6. Let \( I \) denote the closed unit interval \([0, 1]\) in \( \mathbb{R} \) (the real numbers with its usual topology). Prove or disprove that in \( \mathbb{R}^\mathbb{R} \) with its usual product topology \((I^\circ)^\mathbb{R} = (I^\circ)^\circ\), (where \( A^\circ \) denotes the interior of \( A \)).

7. Let \( X \xrightarrow{f} Y \) be any continuous function.
   (a) Prove that there is a factorization of \( f \) \( X \xrightarrow{f} Y = X \xrightarrow{q} Z \xrightarrow{m} Y \) where \( q \) is a quotient map and \( m \) is a one-to-one continuous function.
   (b) Prove that the factorization \( f = m \circ q \) of part (a) is essentially unique in the sense that if \( X \xrightarrow{f} Y = X \xrightarrow{\hat{q}} \hat{Z} \xrightarrow{\hat{m}} Y \) is also a factorization of \( f \) with \( \hat{q} \) a quotient map and \( \hat{m} \) one-to-one and continuous, then there is a unique homeomorphism \( Z \xrightarrow{h} \hat{Z} \) such that \( h \circ q = \hat{q} \) and \( \hat{m} \circ h = m \).
8. Prove that a filter $F$ on a space $X$ converges to a point $x \in X$ if and only if the net $f$ based on $F$ converges to $x$.

9. Let $(X, \leq)$ be a linearly ordered set and let $\tau$ be the topology on $X$ inherited from the order. If $A$ is a subset of $X$, there are two natural ways to topologize $A$. The first, $\tau_1$, is as a subspace of $(X, \tau)$. The second, $\tau_2$, is as an ordered space with the order on $A$ inherited from $(X, \leq)$. Prove or disprove that in all cases $(A, \tau_1) = (A, \tau_2)$.

10. Let $A$ be a connected subset of a connected space $X$, and let $C$ be a component of $X - A$. Prove $X - C$ is connected.

11. Prove that a Hausdorff space with a basis consisting of sets that are both open and closed is totally disconnected.

12. Prove or disprove.
   (a) Every quotient map is an open map.
   (b) Every open map is a quotient map.

13. Prove that a compact subset of a Hausdorff space is closed.

14. Find a flaw in the following purported proof of the statement:

   *Every discrete subspace of a topological space that has no isolated points is nowhere dense.*

   Purported proof:

   Suppose $X$ is a space with no isolated points and $D$ is a discrete subspace of $X$. If $D$ is not nowhere dense, then there is a nonempty open set $U$ with $U \subseteq \overline{D}$. Thus there is some $d \in D \cap U$. Since $D$ is discrete, there is some open set $W \subseteq U$ such that $W \cap D = \{d\}$. Since $X$ has no isolated points $W \neq \{d\}$. Thus $W \setminus \{d\}$ is a nonempty open set contained in $\overline{D}$. Thus there is some $\tilde{d} \in (W \setminus \{d\}) \cap D \subseteq W \cap D = \{d\}$, which is a contradiction. Thus $D$ must be nowhere dense. 

15. Assume $X$ is a normal space. Let $\beta(X)$ be the Čech-Stone compactification of $X$, and let $y \in \beta(X) - X$. Prove that $y$ is not the limit of a sequence of points of $X$.