Choose and work any 6 of the following 15 problems. Start each problem on a new sheet of paper. Do not turn in more than six problems. In the problems below, a space always means a topological space.

1. Prove or disprove: For any space $X$ with topology $\mathcal{J}$, the family $\mathcal{B} = \{ A \subseteq X \mid A$ equals the interior of its closure $\}$ forms a base for some topology $\mathcal{J}'$ on $X$.

2. Prove that a quotient space of a locally connected space is locally connected.

3. Let $f : X \to Y$ be an open map between the spaces $X$ and $Y$. Let $B \subseteq Y$ and $A = f^{-1}[B]$. Prove that the restriction $\overline{f} : A \to B$ (i.e., $\overline{f}(a) = f(a)$) is an open map from $A$ to $B$.

4. (a) State the Axiom of Choice.

(b) State the Well-Ordering Theorem.

(c) Either use the Axiom of Choice to prove the Well-Ordering Theorem or use the Well-Ordering Theorem to prove the Axiom of Choice.

5. Prove that the plane $\mathbb{R}^2$ with its usual topology is not equal to a countable union of straight lines.

6. Prove that the Sorgenfrey line $X = \mathbb{R}$ with basis $\{(a,b) \mid a,b \in \mathbb{R}\}$ is a paracompact space.
7. Let \( f : [a, b] \to \mathbb{R} \) be a real-valued function on a closed interval and let \( G = \{(x, f(x)) \in \mathbb{R} \times \mathbb{R} \mid a \leq x \leq b\} \) be its graph. Prove or give a counterexample for the following.

(a) If \( G \) is connected, then \( f \) is continuous.

(b) If \( f \) is continuous, then \( G \) is connected.

8. Prove that the net based on an ultrafilter is an ultranet.

9. Let \( Y \) be a compact space. Prove that the projection map \( \pi_1 : X \times Y \to X \) is a closed map.

10. Let \( A \) be a subset of a complete metric space \( X \). Prove that \( A \) is totally bounded if and only if the closure \( \overline{A} \) is compact.

11. Let \( X \) be a completely regular \( T_1 \) space (i.e., one point sets are closed, and for each closed set \( C \) and point \( x \not\in C \), there exists a continuous function \( f : X \to [0, 1] \) with \( f(x) = 1 \) and \( f[C] = \{0\} \)). Prove that the Stone-Čech compactification \( \beta(X) \) is connected if and only if \( X \) is connected.

12. Let \( f : X \to Y \) be a continuous surjective map from a compact space \( X \) to a Hausdorff space \( Y \). Prove that \( f \) is a quotient map.

13. Let \( D \) be a dense subset of a metric space \( X \), and let \( Y \) be a complete metric space. Prove that any uniformly continuous function \( f : D \to Y \) can be extended to a uniformly continuous function \( F : X \to Y \) (i.e., \( F\big|_D = f \)).

14. Prove that a metric space is compact if and only if every sequence has a convergent subsequence.

15. Prove that each metric space is a normal space.