Choose and work and 6 of the following problems. Start each new problem on a new sheet of paper. Do not turn in more than six problems. Below a “space” always means a “topological space”.

1. Prove or disprove:
   (a) Closed subspaces of path connected spaces are path connected.
   (b) If \( f : X \to Y \) is continuous and \( X \) is path connected, then \( f[X] \) is path connected.

2. Let \( A \) be a collection of subsets of the topological space \( X \) such that \( X = \bigcup A \). Consider the function \( f : X \to Y \); suppose that \( f|A \) is continuous for each \( A \in A \).
   (a) Show that if \( A \) is finite and each member of \( A \) is closed, then \( f \) is continuous.
   (b) Give an example to show that the word “finite” in part (a) cannot be changed to “countable”.

3. Let \( A \) and \( B \) be disjoint compact subsets in the Hausdorff space \( X \). Show that there are disjoint open subsets \( U \) and \( V \) of \( X \) such that \( A \subseteq U \) and \( B \subseteq V \).

4. Let \( Y \) be an ordered set with the order topology. Let \( f, g : X \to Y \) be continuous.
   (a) Let \( h : X \to Y \) be the function given by
   \[
   h(x) := \min\{f(x), g(x)\}.
   \]
   Show that \( h \) is continuous.
   (b) Show that the set \( \{x \in X | f(x) \leq g(x)\} \) is closed in \( X \).

5. Let \( X \) be a complete metric space and \( f : X \to \mathbb{R} \) a continuous real-valued function on \( X \). Show that every nonempty open subset of \( X \) contains a nonempty open subset on which \( f \) is bounded.

6. Let \( f : X \to Y \) be a continuous surjective map, where \( X \) is compact and \( Y \) is Hausdorff. Show that \( f \) is a quotient map.

7. A space \( X \) is said to be completely regular if one-point sets are closed and if for each point \( x_0 \) and each closed subset \( A \) not containing \( x_0 \), there is a continuous function \( f : X \to [0, 1] \) such that \( f(x_0) = 1 \) and \( f[A] \subset \{0\} \).
   Show that every locally compact Hausdorff space is completely regular.

8. If \( f : X \to Y \) and \( g : Y \to X \) are continuous functions such that \( g \circ f \) is the identity function on \( X \), prove that \( f \) is a topological embedding and that \( g \) is a quotient map.

9. If \( f \) and \( g \) are real-valued continuous functions with the same domain, prove that \( f + g \) is continuous, where \( (f + g)(x) \equiv f(x) + g(x) \) for any \( x \) in the domain.

10. Prove that a filter \( G \) on a set \( X \) is an ultrafilter if and only if for each subset \( A \) of \( X \), either \( A \in G \) or \( X \setminus A \in G \).

11. Prove or disprove:
   (a) Every compact subset of a Hausdorff space is closed.
(b) Every closed subset of a Hausdorff space is compact.

12. Show that a metrizable space $X$ has a countable dense subset if and only if it has a countable basis.

13. Prove or disprove that closed subspaces of normal spaces are normal.

14. Let $Y$ be a metric space and let $f_n : X \to Y$ be a sequence of continuous functions and $f : X \to Y$ a (not necessarily continuous) function. Suppose that $\{f_n\}$ is equicontinuous and $f_n(x) \to f(x)$ for each $x \in X$ (point-wise convergence). Show that $f$ is continuous.