1. Prove that the interval $[0,1]$ with the usual topology is connected.

2. Prove or disprove: Every compact Hausdorff space is separable.

3. Let $\pi_1 : X \times Y \to X$ be the projection map, and let $Y$ be compact. Prove $\pi_1$ is a closed map (where $X \times Y$ has the product topology).

4. Prove that a paracompact Hausdorff space is regular.

5. Use Zorn’s lemma to prove that for every set $X$ and relation $R$ there is maximal $A \subseteq X$ such that $A \times A \subseteq R$.

6. Prove that the plane $\mathbb{R}^2$ is not a countable union of straight lines.

7. If $\Omega$ is the first uncountable ordinal, prove that $[0, \Omega]$ with the order topology is compact.

8. Prove that a connected normal Hausdorff space containing more than one point is uncountable.

9. Let $A$ be a connected subset of a connected space $X$, and let $C$ be a component of $X - A$. Prove that $X - C$ is connected.

10. Let $C(X,Y)$ be the set of continuous functions from $X$ to $Y$, given the compact-open topology. Let $e : C(X,Y) \times X \to Y$ be the evaluation map $e(f,x) = f(x)$. Prove that if $X$ is locally compact Hausdorff, then $e$ is continuous.

11. Let $f : X \to Y$ be a continuous surjective map from a compact space $X$ to a Hausdorff space $Y$. Prove that $f$ is a quotient map.

12. Find an incorrect statement in the proof of the following theorem and prove that it is an incorrect statement.

**Theorem.** If $B^2 = \{(x_1, x_2) \in \mathbb{R}^2| x_1^2 + x_2^2 \leq 1 \}$ has the usual topology, then each continuous function $f : B^2 \to B^2$ has a fixed point.

**Proof:** Suppose that $f : B^2 \to B^2$ is a continuous function with no fixed points. Let $\pi_1 : B^2 \times B^2 \to B^2$ be first projection, let

$$\Delta = \{(z,z)| z \in B^2\}$$

be the diagonal in $B^2 \times B^2$, and let

$$F = \{(z, f(z)| z \in B^2\}$$

be the graph of $f$. Since $f$ is continuous, $\pi_1|_F : F \to B^2$ is a homeomorphism. Since $B^2$ is connected, $F$ is therefore connected. However,

$$F \subseteq B^2 \times B^2 - \Delta$$

which is a contradiction since $B^2 \times B^2 - \Delta$ is disconnected.

13. Prove that the topologist’s comb

$$C = \left( \bigcup_{n=1}^{\infty} \frac{1}{n} \times [0,1] \right) \cup ([0,1] \times \{0\}) \cup (\{0\} \times [0,1])$$

is not a retract of the square $S = [0,1] \times [0,1]$ (with both $C$ and $S$ having the usual Euclidean subspace topologies).