Work 6 of the following problems. Start each problem on a new sheet of paper. Do not turn in more than 6 problems. Assume that all products have the product topology.

1. Prove that $[0, 1]$, with the usual topology, is connected.

2. Prove that if $A$ is a retract of a Hausdorff space $X$, then $A$ is closed in $X$.

3. Prove that a quotient of a locally connected space is locally connected.

4. Let $f : X \to Y$ be an open, continuous surjection. Prove that $Y$ is Hausdorff if and only if the set
   \[ C = \{ (x_1, x_2) \in X \times X | f(x_1) = f(x_2) \} \]
   is closed subset of $X \times X$.

5. Let $S^1$ have the usual topology. Prove that $(\mathbb{Q} \times \mathbb{Q}) \cap S^1$ is dense in $S^1$.

6. Let $E$ denote the set of real numbers with the Sorgenfrey topology, which has basis consisting of all half-open intervals of the form $[x, y)$. Prove that any compact subset of $E$ is countable.

7. Prove that the first projection $\pi_1 : X \times Y \to X$ is closed, if $Y$ is compact.

8. Let $\mathbb{R}^2$ have the usual topology. Prove that if $U$ is a convex open subset of $\mathbb{R}^2$, then
   \[ \overline{U^0} = U, \]
   where “$\overline{\phantom{0}}$” indicates closure and “$^0$” indicates interior.

9. Let $X$ be a compact Hausdorff space. Prove that if every point of $X$ is a limit point of $X$, then $X$ is uncountable.

10. State the Axiom of Choice and Zorn’s Lemma, and prove that Zorn’s Lemma implies the Axiom of Choice.

11. Let $X$ be a metric space. Show that if every family of pairwise disjoint non-empty open subsets of $X$ is countable, then $X$ is separable.

12. A space is called functionally Hausdorff if for every pair of distinct points $x$ and $y$ in $X$, there exists a continuous function $f : X \to [0, 1]$ with $f(x) = 0$ and $f(y) = 1$. Either prove or disprove that every product of functionally Hausdorff spaces is functionally Hausdorff.