1) Suppose \( U = \{ U_j \mid j \in J \} \) is a finite open cover of a normal topological space \( X \). Give a direct construction of an open cover \( \{ V_j \mid j \in J \} \) so that \( \overline{U_j} \subseteq V_j \) for each \( j \in J \). Consider a partition of unity subordinate to \( U \).

2) Prove that a metric space is compact if and only if it is sequentially compact.

3) Prove that the one-point compactification of a 2nd countable, locally compact space is metrizable.

4) Suppose that \( f : [0,1] \rightarrow S^2 = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1 \} \) is continuous and satisfies
   a) \( f(0) = f(1) \)
   b) \( f([0,1]) \) is one-to-one.

   Show that
   a) \( \text{image } f \cong S^2 \)
   b) There is a continuous homotopy \( F : [0,1] \times [0,1] \rightarrow S^2 \)

5) A topological space is said to be extremely disconnected if the closure of every open set is open.
   a) Show that a space is extremely disconnected if and only if every two disjoint open sets have disjoint closures.
   b) Let \( N \) be the natural numbers with the discrete topology and \( X \) be the Stone–Čech compactification of \( N \). Show that \( X \) is extremely disconnected.
7) Let \( S^n = \{ (x_1, x_2, ..., x_{n+1}) \mid \sum_{i=1}^{n+1} x_i^2 = 1 \} \subseteq \mathbb{R}^{n+1} \) be \( n \geq 1 \).

Define an equivalence relation \( ~ \) on \( S^n \) by \( x ~ y \) if \( x = \frac{y}{|y|} \) and let \( P^n \) be the topological space \( S^n/\sim \).

Let \( D^n = \{ (x_1, x_2, ..., x_n) \mid \sum_{i=1}^{n} x_i^2 = 1 \} \). Then \( S^n \) is a closed subset of \( D^n \). Let \( \pi_n : S^n \rightarrow P^n \) be the natural projection. Show that \( \pi_n \) is homeomorphic to \( P^n \) for \( n \geq 2 \).

3) Show that \( \{ (x, \sin 1/x) \mid 0 < x \leq 1 \} \cup \{ (0, y) \mid -1 \leq y \leq 1 \} \subseteq \mathbb{R}^2 \) is connected but not path connected.

5) Define \( f : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{R} \) by \( f(x, y) = \sqrt{x^2 + y^2} \).

Show that \( f \) is continuous.

10) Let \( X \) be a compact and Hausdorff. Let \( \mathcal{F} = \{ f : X \rightarrow Y \mid f \text{ continuous} \} \) and \( Y \subseteq X \) be fixed. Show that \( \mathcal{F} \) is a compact subset of \( C(X, X) \) with respect to the compact-open topology.