Answer all eight questions. Throughout, \((X, \mathcal{M}, \mu)\) denotes a measure space, \(\mu\) denotes a positive measure unless otherwise specified, and all functions are assumed to be measurable.

1. Prove that \(L^p(\mu)\) is complete for \(1 \leq p \leq \infty\).

2. (a) Let \((X, \mathcal{M}, \mu)\) be a measure space. Define what it means for \(\mu\) to be \(\sigma\)-finite.

   (b) Let \((X, \mathcal{M}, \mu)\) and \((Y, \mathcal{A}, \nu)\) be measure spaces. Show by example that we may have
   
   \[
   \int_X \int_Y f(x, y) d\nu(y) d\mu(x) \neq \int_Y \int_X f(x, y) d\mu(x) d\nu(y)
   \]
   
   for some \(f \geq 0\) which is \(\mathcal{M} \times \mathcal{A}\) measurable if we do not assume both \(\mu\) and \(\nu\) are \(\sigma\)-finite.

3. Let \(\lambda\) denote Lebesgue measure on \(\mathbb{R}^n\). Let \(U \subseteq \mathbb{R}^n\) be open. Is it true that \(\lambda(U \setminus U) = 0\)?

4. Prove that the closed unit ball of \(\ell^p(\mathbb{N})\), \(1 \leq p \leq \infty\), is not compact.

5. Suppose \(H\) is a Hilbert space, \(\{e_1, \ldots, e_n\}\) an orthonormal set in \(H\) and \(f \in H\). Prove that the quantity \(\|f - \sum_{j=1}^n a_j e_j\|\) is minimized by taking \(a_j = \langle f, e_j \rangle\) for every \(j\).

6. Let \(w : X \to [0, \infty)\) be measurable, and let \(v(E) = \int_E w d\mu\) for \(E \in \mathcal{M}\).

   Prove: (a) \(v\) is a measure on \(\mathcal{M}\) and (b) \(\int f dv = \int f w d\mu\) for each nonnegative measurable function \(f\) on \(X\).

7. Suppose \(1 \leq p \leq \infty\), \(f \in L^1(\mathbb{R})\) and \(g \in L^p(\mathbb{R})\). Prove \(\|f * g\|_p \leq \|f\|_1 \|g\|_p\).

8. Let \(E \subset \mathbb{R}\) be a Borel set. Prove that \(E' = \{(x, y) \in \mathbb{R}^2 : x + y \in E\}\) is a Borel set.