In problems 1 through 4, \((X, M, \mu)\) is a measure space. In problems 9 and 10, \(|E|\) denotes the Lebesgue measure of the measurable set \(E\).

1. Let \(w : X \to [0, \infty)\) be measurable, and let \(v(E) = \int_E w \, d\mu\) for \(E \in M\).
   
   Prove: (a) \(v\) is a measure on \(M\) and (b) \(\int f \, dv = \int f \, w \, d\mu\) for each nonnegative measurable function \(f\) on \(X\).

2. Suppose \(0 < p < \infty\), \((f_n)\) is a sequence of measurable functions on \(X\), \(f_n \to f\) a.e. and \(\|f_n\|_p \to \|f\|_p < \infty\). Prove \(\|f_n - f\|_p \to 0\).
   
   Hint: Find \(\alpha > 0\) s.t. \(|f_n - f|^p \leq \alpha (|f_n|^p + |f|^p)\) and apply Fatou’s lemma.

3. Let \(I \subseteq \mathbb{R}\) be an open interval, and let \(f\) be a function on \(X \times I\) such that
   
   (i) \(\forall t \in I, f(\cdot, t) \in L_1(\mu)\), and
   
   (ii) \(\exists g \in L_1(\mu)\) such that \(\left| \frac{\partial f}{\partial t}(x, t) \right| \leq g(x)\ \forall (x, t) \in X \times I\).

   Prove:
   
   \[ \frac{d}{dt} \int f(x, t) \, d\mu(x) = \int \frac{\partial f}{\partial t}(x, t) \, d\mu(x). \]

4. Suppose \(\mu(X) < \infty\). A family \(\mathcal{F}\) of measurable functions on \(X\) is said to be uniformly integrable if given \(\varepsilon > 0\), \(\exists \delta > 0\) such that \(E \in M, \mu(E) < \delta\) and \(f \in \mathcal{F} \Rightarrow \int_E |f| d\mu < \varepsilon\).

   Prove Vitali’s theorem: Suppose
   
   (i) \(\{f_n\} \subset L_1(\mu)\) is uniformly integrable,
   
   (ii) \(f_n \to f\) a.e. for some function \(f\), and
   
   (iii) \(|f| < \infty\) a.e.

   Thus \(f \in L_1(\mu)\) and \(\|f_n - f\|_1 \to 0\).

5. Show that for \(1 \leq p \leq \infty\), the closed unit ball of \(\ell_p(\mathbb{N})\) is not compact.

6. Let \(1 \leq p \leq \infty\), \(f \in (\mathbb{R})\) and \(g \in L_p(\mathbb{R})\). Prove \(\|f \ast g\|_p \leq \|f\|_1 \cdot \|g\|_p\).

7. Suppose \(f \in L_2(\mathbb{R}^+)\). Prove that

   \[ x^{-\frac{1}{2}} \int_0^x f(t) \, dt \to 0 \text{ as } x \to 0^+. \]

8. Fix a Lebesgue measurable function \(f\) on \(\mathbb{R}^+\) and define

   \[ \phi(p) = \left( \int_0^\infty |f(x)|^p e^{-x} \, dx \right)^{\frac{1}{p}} \quad (0 < p < \infty). \]

   Prove: (a) \(p < q \Rightarrow \phi(p) \leq \phi(q)\).
(b) If $\phi(p) = \phi(q) < \infty$ for some $0 < p < q < \infty$, then $|f| = \text{const}$ a.e.

9. (a) Fix $0 < \varepsilon < 1$. Construct a closed set $K \subset [0,1]$ such that $|K| > 1 - \varepsilon$ and $K$ contains no rationals.

(b) Does there exist a Borel set $E \subset [0,1]$ such that $0 < |E \cap I| < |I|$ for each nonempty open interval $I \subset [0,1]$?

10. Let $E \subset \mathbb{R}$ be a Lebesgue measurable set such that

   (i) $(E + k) \cap E = \emptyset \quad \forall k \in \mathbb{Z} \setminus \{0\}$ and (ii) $E + \mathbb{Z} = \mathbb{R}$.

   Prove that $|E| = 1$. Hint: Let $I = (0,1)$. Then $I = \bigcup_k I \cup (E + k)$ and $E = \bigcup_k (I + k) \cup E$. 

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