In this exam $\lambda$ denotes Lebesgue outer measure on $\mathbb{R}$ and $(X, \mathcal{A}, \mu)$ denotes a measure space.

1. (a) Define Lebesgue outer measure $\lambda$ on $\mathbb{R}$.
   (b) What does it mean to say that a subset $E$ of $\mathbb{R}$ is $\lambda$-measurable?
   (c) Use the fact that the family $\mathcal{M}_\lambda$ of all $\lambda$-measurable subsets of $\mathbb{R}$ is a $\sigma$-algebra to prove that every Borel subset of $\mathbb{R}$ is $\lambda$-measurable.

2. Prove the following assertions. You may use the fact that the restriction of $\lambda$ to $\mathcal{M}_\lambda$ is countably additive if you wish.
   (a) If $a < b$ in $\mathbb{R}$, then $\lambda([a, b]) = b - a$.
   (b) There exists a compact set $K \subset [0, 1]$ such that $\lambda(K) > \frac{3}{4}$ and $K$ contains no rational number.

4. (a) What does it mean to say that a function $f : X \to [-\infty, \infty]$ is $\mathcal{A}$-measurable?
   (b) Suppose that $f_n : X \to [-\infty, \infty]$ is $\mathcal{A}$-measurable for $n = 1, 2, \ldots$ and define $f$ on $X$ by
   $$f(x) = \lim_{n \to \infty} f_n(x)$$
   for $x \in X$. Use your definition in (a) to prove that $f$ is $\mathcal{A}$-measurable.

5. Let $f : X \to [0, \infty]$ be $\mathcal{A}$-measurable. Define
   $$A = \{(x, y) \in X \times \mathbb{R} : 0 \leq y < f(x)\}$$
   Prove that $A \in \mathcal{A} \times \mathcal{M}_\lambda$ and that if $\mu$ is $\sigma$-finite, then
   $$\mu \times \lambda(A) = \int_X f \, d\mu.$$

6. Let $(f_n)_{n=1}^\infty$ be a sequence of real-valued $\mathcal{A}$-measurable functions on $X$ that converges to some function $f$ at each point of $X$. Prove that for each $a \in \mathbb{R}$ we have
   $$\mu(\{f > a\}) \leq \lim_{n \to \infty} \mu(\{f_n > a\}).$$

7. Let $f : X \to [0, \infty]$ be $\mathcal{A}$-measurable and suppose that $\mu$ is $\sigma$-finite. Define $m(t) = \mu(\{f > t\})$ for each $t \geq 0$. Prove that if $0 < p < \infty$, then
   $$\int_X f^p \, d\mu = p \int_0^\infty t^{p-1} m(t) \, dt.$$  
   [Hint: $f^p(x) = \int_0^{f(x)} pt^{p-1} \, dt$.]

8. Prove the completeness of $L^p(\mu)$ for $1 \leq p < \infty$.

9. Let $2 \leq p < \infty$ and let $f, g \in L^p(\mu)$. Prove that
   $$\| \frac{f + g}{2} \|_p^p + \| \frac{f - g}{2} \|_p^p \leq \frac{1}{2}(\| f \|_p^p + \| g \|_p^p).$$
   [Hints: First show that if $a, b \geq 0$, then $a^p + b^p \leq (a^2 + b^2)^{p/2}$ and $(\frac{a^2 + b^2}{2})^{1/2} \leq \left(\frac{a^p + b^p}{2}\right)^{1/p}$.]
10. Suppose $\mu(X) < \infty$ and let $\nu$ be another (positive) measure on $(X, \mathcal{A})$ with $\nu(X) < \infty$. Let $\nu = \nu_a + \nu_s$ be the Lebesgue decomposition of $\nu$ with respect to $\mu$ and let $w : X \to [0, \infty]$ be a Radon-Nikodym derivative of $\nu_a$ with respect to $\mu$. Define $w_0$ on $X$ by $w_0(x) = \min\{w(x), 1\}$. Define $w_0$ on $X$ by $w_0(x) = \min\{w(x), 1\}$.

Prove the following:

(a) For each $\mathcal{A}$-measurable $f : X \to [0, \infty]$ we have

$$\int f w_0 d\mu \leq \min\left\{ \int f d\nu, \int f d\mu \right\}.$$ 

(b) Suppose that to each $\varepsilon > 0$ corresponds some $\mathcal{A}$-measurable $f : X \to [0, \infty]$ such that

$$\left( \int f d\nu \right) \left( \int f^{-1} d\mu \right) < \varepsilon$$

where $f^{-1} = 1/f$. Then $\nu$ and $\mu$ are mutually singular. [Hint for (b): Apply the Schwarz Inequality to $f^{1/2} \cdot f^{-1/2} = 1$ for the measure $w_0 d\mu$.]