Do as much as you can but remember that five correct solutions will count more than ten half-solutions.

Throughout this exam \((X,\mathcal{A},\mu)\) denotes an arbitrary measure space and \(\lambda\) denotes Lebesgue outer measure in the real line \(\mathbb{R}\).

In 1 and 2 discuss the truth value of the lettered assertions. Quote theorems, give proofs or counter examples; etc.

1. Let \(E\) be a \(\lambda\)-measurable subset of \(\mathbb{R}\).

(a) If \(\lambda(E) > 0\), then \(E\) contains a nonvoid open interval.

(b) If \(E\) is closed but contains no rational number, then \(\lambda(E) = 0\).

(c) If \(\lambda(E) = 0\), then \(E\) is of first Baire category in \(\mathbb{R}\).

2. Let \((f_n)_{n=1}^{\infty}\) be a sequence of \(\mu\)-integrable functions on \(X\). This means that each \(f_n\) is complex-valued, \(\lambda\)-measurable, and \(\int |f_n| \, d\mu < \infty\).

(a) If \(\int |f_n| \, d\mu \leq 1\) for all \(n\), then \(\sum_{n=1}^{\infty} f_n \) converges \(\lambda\)-a.e.

(b) If \(\lim_{n \to \infty} f_n(x) = 0\) for all \(x \in X\), then \(\lim_{n \to \infty} \int f_n \, d\mu = 0\).

(c) If \(\lim_{n \to \infty} \int |f_n| \, d\mu = 0\), then \(\lim_{n \to \infty} f_n = 0\) \(\mu\)-a.e.

3. Prove that if \((f_n)_{n=1}^{\infty}\) is a sequence of \(\mu\)-integrable functions on \(X\) such that

\[
\lim_{m,n \to \infty} \int |f_m - f_n| \, d\mu = 0,
\]

then there exist a subsequence \((f_{n_k})_{k=1}^{\infty}\) and a \(\mu\)-integrable function \(f\) such that

(a) \(\lim_{k \to \infty} f_{n_k}(x) = f(x)\) \(\mu\)-a.e.

and

(b) \(\lim_{n \to \infty} \int |f - f_n| \, d\mu = 0\).
4. Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be Lebesgue integrable. Suppose that
\[
\int_{-\infty}^{\infty} f(x)g(x)\,dx = 0
\]
for each continuous \( g : \mathbb{R} \rightarrow \mathbb{R} \) such that
\[\{x \in \mathbb{R} : g(x) \neq 0\}\] is bounded. Give a detailed proof that \( f = 0 \) almost everywhere.

5. Prove that there exist Banach spaces that are not reflexive.

6. Prove that if \( f \in L_1(\mathbb{R}) \) and \( g \) is defined on \( \mathbb{R} \) by
\[
g(x) = \int_{-\infty}^{\infty} f(t)e^{\lambda tx}\,dt,
\]
then
(a) \( g \) is continuous on \( \mathbb{R} \) and
(b) \( \lim_{|x| \to \infty} g(x) = 0. \)

[Hint: Show that the set of \( f \) for which (a) and (b) hold is both dense and closed in \( L_1 \).

7. Let \( a < b \) be real numbers and let \( f \in L_1([a,b]) \). Define
\[
F(x) = \int_{a}^{x} f(t)\,dt \quad (a \leq x \leq b);
\]
and
\[
V = \sup_{n} \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})|
\]
where this supremum is taken over all (finite) subdivisions \( \{a = x_0 < x_1 < \ldots < x_n = b\} \) of \([a,b]\). Prove that
\[
V = \int_{a}^{b} |f(t)|\,dt.
\]

8. For \( f \in C([a,b]) \) and \( n \in \mathbb{N} \) define
\[
J_n(f) = \int_{a}^{b} f(t) \frac{\sin nt}{t}\,dt.
\]
Prove that there exists such an \( f \) such that
\[
\lim_{n \to \infty} |J_n(f)| = \infty.
\]
9. Let \( a < b \) be real numbers and let \( \phi : [a, b] \to \mathbb{R} \) be Borel measurable and satisfy
\[
\lambda(E) = 0 \implies \lambda(\phi^{-1}(E)) = 0.
\]
(a) Prove that there is a Lebesgue integrable function \( w : \mathbb{R} \to \mathbb{R} \) such that
\[
\int_a^b f(\phi(t)) \, dt = \int_{-\infty}^{\infty} f(x) w(x) \, dx
\]
for all bounded Borel measurable functions \( f : \mathbb{R} \to \mathbb{C} \).
(b) If \( [a, b] = [0, 2\pi] \) and \( \phi(t) = \cos t \), then what is \( w \)?

10. Let \( f, g \in L_1([0, 1]) \). Prove that the formula
\[
h(x) = \int_0^x f(x - t) g(t) \, dt
\]
defines \( h \) almost everywhere on \([0, 1]\), that \( h \in L_1([0, 1]) \), and that
\[
||h||_1 \leq ||f||_1 \cdot ||g||_1
\]
where \( ||\phi||_1 = \int_0^1 |\phi(t)| \, dt \) for \( \phi \in L_1([0, 1]) \).