Real Analysis Qualifying Exam

January, 2002

Instructions: Below you will find 8 problems. Each problem is worth 10 points. Only the best 6 scores will be added.
Time: 2 hours.

Notations: \( \mathbb{R} \) = set of all real numbers; \( \lambda \) = Lebesgue measure on \( \mathbb{R} \); 
\( L^p(I, d\lambda) \) = the space of real-valued Lebesgue measurable functions on \( I \) with \( \int_I |f|^p d\lambda < \infty \) (\( I = \) any interval, \( p > 0 \)).

1. Give an example of a set \( A \subset \mathbb{R} \), which is not Lebesgue measurable.

2. Let \( K \) be a compact Hausdorff space, and let \( (f_n)_{n=1}^\infty \) be a sequence of continuous real-valued functions on \( K \), such that for every \( x \in K \) one has:
   (i) \( f_1(x) \geq f_2(x) \geq f_3(x) \geq \ldots \)
   (ii) \( \lim_{n \to \infty} f_n(x) = 0. \)

   Prove that \( \lim_{n \to \infty} f_n = 0 \) uniformly.

3. Suppose \( (f_n)_{n=1}^\infty \) is a sequence in \( L^1(\mathbb{R}, d\lambda) \), such that
   \[
   \lim_{n \to \infty} \int_\mathbb{R} |f_n| \, d\lambda = 0. \quad (\ast)
   \]

   (a) Prove that there exists a subsequence \( (f_{n_k})_{k=1}^\infty \) of \( (f_n)_{n=1}^\infty \), with \( \lim_{k \to \infty} f_{n_k} = 0 \), a.e.

   (b) Given an example of a sequence \( (f_n)_{n=1}^\infty \), satisfying \( (\ast) \), but which is not convergent a.e. to 0.
4. Let $C[0, 1]$ denote the Banach space of all real-valued continuous functions on $[0, 1]$. (The norm on $C[0, 1]$ is defined by $\|f\| = \max_{t \in [0,1]} |f(t)|$.)

(a) Prove that any finite dimensional linear subspace $V$ of $C[0, 1]$ is closed in the norm topology.

(b) Prove that if $V$ is a linear subspace of $C[0, 1]$, which has a countable infinite linear basis, then $V$ is not closed in the norm topology. (Hint: Use Baire’s Theorem.)

5. Let $f \in L^2([-\pi, \pi], d\lambda)$ be a function with the property:

\[ \int_{-\pi}^{\pi} f(x) \sin nx \, dx = 0, \quad \forall \, n \in \mathbb{N}. \]

Prove that $f(-x) = f(x)$, a.e.

6. Let $f \in L^1(\mathbb{R}, d\lambda)$, and let $(f_n)_{n=1}^{\infty}$ be a sequence in $L^1(\mathbb{R}, d\lambda)$, with

(i) $\lim_{n \to \infty} f_n = f$, a.e.

(ii) $\lim_{n \to \infty} \|f_n\|_1 = \|f\|_1$.

Prove that $\lim_{n \to \infty} \|f_n - f\|_1 = 0$.

7. Let $f \in L^1(\mathbb{R}, d\lambda)$. Prove that

\[ \lim_{n \to \infty} \int_{\mathbb{R}} f(x) \sin nx \, dx = 0. \]

8. Let $(X, \mathcal{M}, \mu)$ be a $\sigma$-finite measure space, and let $\{E_\lambda\}_{\lambda \in \Lambda} \subset \mathcal{M}$ be a disjoint collection. Prove that the set

\[ S = \{ \lambda \in \Lambda : \mu(E_\lambda) > 0 \} \]

is at most countable.