REAL ANALYSIS QUALIFYING EXAM
Fall 1999
(Saeki & Moore)

Answer all eight questions. Throughout, \((X, \mathcal{M}, \mu)\) denotes a measure space, \(\mu\) denotes a positive measure unless otherwise specified, and all functions are assumed to be measurable.

1. Suppose \(1 \leq p \leq \infty\). Show that the closed unit ball of \(\ell_p(\mathbb{N})\) is not compact.

2. Let \(1 \leq p \leq 2\) and \(f \in L^p([0, \infty])\). For \(x \geq 0\) set \(g(x) = \int_x^{x^2} f(t)dt\). Show that \(\lim_{x \to \infty} \frac{g(x)}{x} = 0\).

3. Suppose \(\{f_n\}\) is a sequence of nonnegative measurable functions on \(X\) such that \(\lim_{n \to \infty} f_n(x) = f(x)\) exists a.e. and
\[
\lim_{n \to \infty} \int_X f_n d\mu = \int_X f d\mu < \infty.
\]
Prove that \(\lim_{n \to \infty} \int_E f_n d\mu = \int_E f d\mu\) for every measurable set \(E \subseteq X\).

4. Prove that if \(A\) and \(B\) are Borel sets in topological spaces \(X\) and \(Y\) respectively, then \(A \times B\) is a Borel set in \(X \times Y\).

5. Let \(f \in L^1(\mu)\). Prove that given \(\varepsilon > 0\) there exists \(\delta > 0\) such that \(\left| \int_E f d\mu \right| < \varepsilon\) whenever \(\mu(E) < \delta\).

6. Prove that \(L^p(\mu)\) is complete for \(1 \leq p \leq \infty\).

7. Assume now that \((X, \mathcal{M}, \mu)\) is a \(\alpha\)-finite measure space. Let \(f : x \to [0, \infty),\ 0 < p < \infty\). Show \(\int_X f^p d\mu = \int^1 \int f(x, t)^p \mu(dx) dt\).

8. Suppose \(f\) is defined on \(X \times (0, 1)\) and that for each fixed \(t \in (0, 1),\ f(0, t) \in L^1(\mu)\). Suppose also that \(\frac{\partial f}{\partial t}(x, t)\) exists for every \((x, t) \in X \times (0, 1)\) and \(\frac{\partial f}{\partial t}\) is bounded on \(X \times (0, 1)\). Show that
\[
\frac{d}{dt} \int_X f(x, t)d\mu(x) = \int_X \frac{\partial f}{\partial t}(x, t)d\mu(x).
\]