REAL ANALYSIS QUALIFYING EXAM
Fall 1998
(Saeki & Vaninsky)

Do all of the problems below.

Let \((X, \mathcal{A}, \mu)\) be a measure space.

1. Let \(\varphi : X \to \Omega\), where \(\Omega\) is a topological space.

   (a) What does it meant that \(\varphi\) is measurable?

   (b) Suppose \(\varphi\) is measurable and \(E\) is a Borel subset of \(\Omega\). Prove that \(\varphi^{-1}(E)\) is measurable.

2. Let \(f \in L^1(\mu)\). Prove that \(|\int f \, d\mu| \leq \int |f| \, d\mu\).

3. Let \(1 \leq p < \infty\). Prove the completeness of \(L^p(\mu)\).

4. Prove the Dominated Convergence Theorem by applying Fatou’s Lemma.
5. Let \( \varphi : X \to \Omega \) be measurable, where \( \Omega \) is a topological space. Define \( \nu(E) := \mu(\varphi^{-1}(E)) \) for each \( E \in \mathcal{B}_\Omega \) (the Borel subsets of \( \Omega \)). Prove:

(a) \( \nu \) is a Borel measure.

(b) Let \( f : \Omega \to [0, \infty] \) be Borel measurable. Then
\[
\int_\Omega f \, d\nu = \int_X f \circ \varphi \, d\mu.
\]

6. Let \( f : X \times [0, 1] \to \mathbb{C} \). State (nontrivial) conditions on \( f \) that guarantee
\[
(\ast) \quad \frac{d}{dt} \int f(x, t) \, d\mu(x) = \int \frac{\partial f}{\partial t}(x, t) \, d\mu(x) \quad \forall \, t \in (0, 1)
\]
and then prove \((\ast)\).

7. Let \( L^\infty(\mu) + L^1(\mu) = \{g + h : g \in L^\infty(\mu), \ h \in L^1(\mu)\} \). Prove that
\[
L^p(\mu) \subset L^\infty(\mu) + L^1(\mu) \quad \forall \, p \in [1, \infty].
\]

8. Let \( \varepsilon > 0 \). Construct a compact set \( K \subset [0, 1] \mathbb{Q} \) such that \( |K| > 1 - \varepsilon \), where \( |K| \) is the Lebesgue measure of \( \kappa \). 