1. Define Lebesgue outer measure, \( \lambda^* \), on \([0,1]\).

2. Let \((X, S, \mu)\) be a finite measure space and suppose that \(\{f_n\}\) is a sequence of measurable functions which converges to a finite function \(f\) a.e.. Let \(\varepsilon > 0\) and set \(A_N = \{x : \sup_{n \geq N} |f_n(x) - f(x)| > \varepsilon\}\). Prove that \(\mu(A_N) \to 0\) as \(N \to \infty\).

3. Suppose \(f \in L^p([0,\infty))\) where \(1 \leq p \leq 2\). For \(x \geq 0\) set \(g(x) = \int_x^{\infty} f(t)dt\). Show that \(\lim_{x \to \infty} \frac{g(x)}{x} = 0\).

4. Let \(f\) be a \(C^\infty\) function (that is, \(f\) and all its derivatives exist and are continuous) from \(\mathbb{R}\) to \(\mathbb{R}\) with the property that for each \(x \in \mathbb{R}\) there exists a \(k \in \mathbb{N}\) (depending on \(x\)) such that \(\frac{\partial^k f}{\partial x^k}(x) = 0\). Show that there exists an interval \(I \subseteq \mathbb{R}\), \(I \neq \emptyset\) such that \(f\big|_I\) is a polynomial.

5. (a) Construct a bounded, Lebesgue integrable function \(g(x)\) on \([0,1]\) such that \(\int_0^1 |f(x) - g(x)|dx > 0\) for every Riemann integrable function \(f(x)\) on \([0,1]\).

   (b) Can you construct such a \(g(x)\) so that \(\int_0^1 |f(x) - g(x)|dx > 10^{-5}\) for every Riemann integrable function \(f(x)\) on \([0,1]\)?

6. Suppose \(f\) is a Lebesgue measurable function on \([0,1]\). Is it true that if \(f' = 0\) a.e. then \(f\) cannot be strictly increasing?

7. Suppose \(\Phi : [0,\infty) \to [0,\infty)\) is strictly increasing continuously differentiable function with \(\Phi(0) = 0\). Suppose that \(f \in L^1(X, \mathcal{M}, \mu)\) and for \(\lambda > 0\) set \(m(\lambda) = \mu(\{x \in X : |f(x)| > \lambda\})\). Prove that \(\int_X \Phi(f(x))d\mu(x) = \int_0^\infty m(\lambda)d\Phi(\lambda)\).

8. Suppose \(\mu\) is a complex measure on \(\mathbb{R}^n\) with the property that \(\int_{\mathbb{R}^n} f(x)d\mu \geq 0\) whenever \(f \geq 0\) is a continuous function on \(\mathbb{R}^n\) with compact support. Show that \(\mu\) is a positive measure.

9. Let \(v, \mu\) be complex measures on \((X, \mathcal{M})\). Suppose \(v \ll \mu\). Show that for every \(\varepsilon > 0\) there exists a \(\delta > 0\) such that if \(|\mu(E)| < \delta\) then \(|v(E)| < \varepsilon\).

10. Suppose \((X, \mathcal{M}, \mu)\) is a measure space and \(\mu\) is positive and \(\sigma\)-finite.

    (a) Suppose \(f \in L^1(X, \mathcal{M}, \mu)\) and suppose \(\mathcal{A} \subseteq \mathcal{M}\) is also a \(\sigma\)-algebra. Prove that there exists a function \(g \in L^1(X, \mathcal{A}, \mu)\) such that \(\int_E gd\mu = \int_E f d\mu\) for all \(E \in \mathcal{A}\).

    (b) If \(g\) and \(f\) are related as in part (a) we write \(g = E(f|\mathcal{A})\). Suppose that \(\mathcal{B} \subseteq \mathcal{A}\) is also a \(\sigma\)-algebra on \(X\). Show that \(E(E(h|\mathcal{A})|\mathcal{B}) = E(h|\mathcal{B})\) whenever \(h \in L^1(X, \mathcal{M}, \mu)\).