1. (a) Give a definition of Lebesgue outer measure $\lambda$ on $\mathbb{R}$.

(b) Use your definition in (a) to prove that $\lambda([0,1]) = 1$.

(c) Prove that if $\varepsilon > 0$, then there is a closed subset $A$ of $[0,1]$ containing no rational number such that $\lambda(A) > 1 - \varepsilon$.

2. Let $f: \mathbb{R} \to \mathbb{C}$ be continuous. Suppose that

$$\int_{\mathbb{R}} f(x)g(x)dx = 0$$

whenever $g: \mathbb{R} \to \mathbb{C}$ is continuous with compact support. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

3. Let $(f_n)_{n=1}^{\infty}$ be a sequence of Lebesgue integrable functions on $\mathbb{R}$ such that

$$\sum_{n=1}^{\infty} \int_{\mathbb{R}} |f_n(x)|dx < \infty.$$ 

Prove that there is a Lebesgue integrable function $f$ on $\mathbb{R}$ such that

(a) $f(x) = \sum_{n=1}^{\infty} f_n(x)$ a.e. on $\mathbb{R}$,

(b) $\lim_{N \to \infty} \int_{\mathbb{R}} |f(x) - \sum_{n=1}^{N} f_n(x)|dx = 0$,

and

(c) $\int_{\mathbb{R}} f(x)dx = \sum_{n=1}^{\infty} \int_{\mathbb{R}} f_n(x)dx$.

4. State the following:

(a) The Riesz representation theorem for nonnegative linear functionals.

(b) Fubini's theorem on multiple integrals.

(c) The Radon-Nikodým Theorem.

(d) The Hahn-Banach Theorem.
5. Let \((X, \mathcal{M})\) be a measurable space and let \(\mu\) and \(\nu\) be finite measures on it. Prove that the following two assertions are equivalent:

(a) \(\mu(E) = 0\) whenever \(E \in \mathcal{M}\) and \(\nu(E) = 0\).

(b) To each \(\varepsilon > 0\) corresponds some \(\delta > 0\) such that \(\mu(E) < \varepsilon\) whenever \(E \in \mathcal{M}\) and \(\nu(E) < \delta\).

6. Let \(X\) be an infinite compact Hausdorff space. Prove that the Banach space \(C(X)\) is not reflexive. [Hint: For an appropriate \(a \in X\), define \(L\) on the space \(M(X)\) of complex regular Borel measures on \(X\) by \(L(\mu) = \mu\{\{a\}\}\).]

7. Let \(f, g \in L_1(\mathbb{IR})\). Prove that the formula

\[
h(x) = \int_{-\infty}^{\infty} f(x - t)g(t)dt
\]

defines \(h\) almost everywhere on \(\mathbb{IR}\), that \(h \in L_1(\mathbb{IR})\), and that \(\|h\|_1 \leq \|f\|_1 \cdot \|g\|_1\).

8. Let \((X, \mathcal{M})\) be a measurable space and let \(f: X \to [0, \infty)\) be \(\mathcal{M}\)-measurable. Prove that there exist sequences \((a_n)_{n=1}^{\infty} \subseteq [0, \infty)\) and \((A_n)_{n=1}^{\infty} \subseteq \mathcal{M}\) such that

\[
f(x) = \sum_{n=1}^{\infty} a_n \xi_{A_n}(x)
\]

for all \(x \in X\). Here \(\xi_B\) is the characteristic function of \(B\).

9. Let \((X, \mathcal{M}, \mu)\) be a finite measure space and let \(F\) be a nonvoid subfamily of \(\mathcal{M}\). Prove that there is a set \(B \in \mathcal{M}\) satisfying both

(a) \(\mu(F \setminus B) = 0\) for all \(F \in F\)

and

(b) if \(A \in \mathcal{M}\) and \(\mu(F \setminus A) = 0\) for all \(F \in F\), then \(\mu(B \setminus A) = 0\).

[Hint: Consider the number \(\beta = \sup\{\mu(U) : U\) is a countable union of members of \(F\).\}]
10. (a) Let \( (n_j)_{j=1}^{\infty} \) be a sequence of integers such that \( |n_1| < |n_2| < \ldots \). Prove that the set \( E \) of all \( x \in \mathbb{R} \) such that
\[
\lim_{j \to \infty} \frac{x}{n_j}
\]
exists must have Lebesgue measure 0. [Hint: Let \( f(x) \) be the above limit if \( x \in E \) and \( f(x) = 0 \) otherwise. Consider the Fourier coefficients of \( f \).]

(b) Prove that if \( n_j = j! \), then \( E \) is uncountable.