REAL ANALYSIS
QUALIFYING EXAM
OCTOBER 22, 1982

You have three hours to solve as many of the following ten problems as you can.

1. a) Define Lebesgue outer measure \( \lambda \) on \( \mathbb{R} \).
Without assuming any properties of \( \lambda \) other than your definition, prove that

b) \( \lambda([0,1]) = 1 \) and

c) there exists a closed set \( F \subseteq [0,1] \) such that \( F \) contains no rational number and \( \lambda(F) > 1/2 \).

2. Let \( (E_n)_{n=1}^{\infty} \) be a sequence of Lebesgue measurable subsets of \([0,1]\) such that \( \lim_{n \to \infty} \lambda(E_n) = 1 \) and let \( 0 < \varepsilon < 1 \).
Prove that there exists a subsequence \( (E_{n_k})_{k=1}^{\infty} \) for which
\[
\lambda\left( \bigcap_{k=1}^{\infty} E_{n_k} \right) > 1 - \varepsilon.
\]

3. Let \( (X, \mathcal{A}, \mu) \) be a measure space with \( \mu(X) = 1 \). Define 

\( \log 0 = -\infty \) and \( \log \infty = \infty \). Let \( f:X \to [0,\infty] \) be \( \mathcal{A} \)-measurable.
Prove that if the integral on the left is defined, then
\[
\int (\log f(x)) \, d\mu(x) \leq \log(\int f(x) \, d\mu(x))
\]
and that equality obtains if and only if \( f \) is \( \mu \)-a.e. equal to a constant. [Hint: Check that \( \log t \leq t - 1 \) and put \( t = f(x)/\int f \, d\mu \).]

4. Let \( f \in L_1(\mathbb{R}) \). Define \( \phi:\mathbb{R} \to \mathbb{C} \) by
\[
\phi(t) = \int_{-\infty}^{\infty} \frac{f(x)}{1 + x^2 t^2} \, dx.
\]
Prove that

a) \( \lim_{|t| \to \infty} |\phi(t)| = 0 \).

b) \( \phi \) is differentiable at each \( t \in \mathbb{R} \), and

c) \( \phi'(t) = -2t \int_{-\infty}^{\infty} \frac{x^2 f(x)}{(1 + x^2 t^2)^2} \, dx \).
5. Let \((f_n)_{n=1}^{\infty}\) be a sequence in \(L_1\) on a measure space \((X, A, \mu)\). Suppose that \(f = \lim f_n\) exists a.e. on \(X\) and that

\[
\|f_n\|_1 \leq C \quad \text{and} \quad \int_X \log|f_n| \, d\mu \geq -C \quad (n = 1, 2, \ldots)
\]

where \(C\) is a finite constant. Prove that both \(f\) and \(\log|f|\) belong to \(L_1\).

Suggestion: Apply Fatou's Lemma to \((|f_n|)_{n=1}^{\infty}\) and \((g_n)_{n=1}^{\infty}\), where \(g_n = |f_n| - \log|f_n|\).

6. a) State Fubini's Theorem.
    b) Let \(f, g \in L_1(\mathbb{R})\). Prove that the formula

\[
h(x) = \int_{-\infty}^{\infty} f(xy)g(y)\sin y \, dy
\]

defines a function \(h\) at almost every \(x \in \mathbb{R}\) and that \(h \in L_1(\mathbb{R})\). [You may presume that the function \((x, y) \mapsto f(xy)\) is measurable.]

7. Let \((X, A, \mu)\) be a measure space, and let \(\phi\) be a real-valued measurable function on \(X\). Define

\[
\nu(B) = \mu(\phi^{-1}(B)) \quad (B \in \mathcal{B}),
\]

where \(\mathcal{B}\) denotes the \(\sigma\)-algebra of all Borel sets in \(\mathbb{R}\). Prove the following:

a) \(\nu\) is a measure on \((\mathbb{R}, \mathcal{B})\).

b) If \(f\) is a non-negative, simple, Borel function on \(\mathbb{R}\), then

\[
\int_{\mathbb{R}} f \, d\nu = \int_X f \circ \phi \, d\mu.
\]

c) The above formula holds for all non-negative Borel functions \(f\) on \(\mathbb{R}\).

8. Let \(\mu\) be a regular Borel measure on the plane \(\mathbb{R}^2\). Define \(\nu\) on the Borel sets \(\mathcal{B}\) of \(\mathbb{R}\) by

\[
\nu(A) = \mu(A \times \mathbb{R}).
\]

Prove that there exists a mapping \(f: \mathcal{B} \rightarrow \mathbb{R}_+\) into \(L_1^+(\nu)\) such that

a) \(\mu(A \times B) = \int_A f_B \, d\nu\)
    and

b) \(f_{B_1} + f_{B_2} = f_{B_1 \cup B_2} + f_{B_1 \cap B_2}\) whenever \(A, B, B_1, B_2 \in \mathcal{B}\).
9. Prove that if \( f : \mathbb{R} \rightarrow \mathbb{C} \) is continuous with \( f(x + 1) = f(x) \) for all \( x \in \mathbb{R} \) and if \( \xi \in \mathbb{R} \) is irrational, then
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} f(j\xi) = \int_{0}^{1} f(x) \, dx.
\]

[Hint: Show first that the set \( F \) of all \( f \) for which the conclusion obtains is a linear space that contains all \( f \) of the form
\[
f(x) = e^{2\pi ikx} \quad (k \in \mathbb{Z}).
\]

10. Prove that \( f \in L_1(\mathbb{R}) \) implies \( \hat{f} \in C_0(\mathbb{R}) \), where
\[
\hat{f}(t) = \int_{-\infty}^{\infty} f(x)e^{itx} \, dx \quad (t \in \mathbb{R}).
\]

[Suggestions: (i) Consider the case where \( f \) is the characteristic function of a bounded interval. (ii) Use the fact that the step functions (in \( L_1(\mathbb{R}) \)) are dense in \( L_1(\mathbb{R}) \).]