NUMERICAL ANALYSIS QUALIFYING EXAM
Fall, 2004

(do at least 3 problems from problems 1-4, and do at least 3 problems from problems 5-8, you may do as many as you can)

(1) Show that in the bisection method solving an equation \( f(x) = 0 \) where \( f \in C([a, b]) \) and \( f(a) \cdot f(b) < 0 \), if one wants an accuracy of \( \epsilon \) in the result, that is the iteration is stopped when \( |a_n - b_n| < \epsilon \), then the number of steps necessary to achieve this is no more than
\[
1 + \frac{\ln(b-a)}{\ln 2}
\]
where the interval \([a, b]\) is the one on which the bisection method applies with \( a_0 = a \), and \( b_0 = b \).

(2) Let \( \alpha \) be a root of multiplicity \( m \) for the equation \( f = 0 \), where \( f \) is sufficiently smooth near \( \alpha \).
Show that if the “multiply-relaxed” Newton method
\[
x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)}
\]
converges to \( \alpha \), it does so at least quadratically.

(3) Suppose that \( k \) and \( n \) are positive integers with \( k < n \) and that \( f \) is a real valued function continuous on the interval \([-1, 2]\). For each integer \( m \geq n \), \( S_m \) is a piecewise polynomial approximation to \( f \) on \([0, 1]\) defined as follows: First, set up a mesh \( \{x_j\}_{j \in \mathbb{Z}} \cap [-1, 2] \) where \( x_j = jh \) with \( h = 1/m \). Then on each subinterval \([x_j, x_{j+1}] \cap [0, 1]\) define \( S_m(x) = p_j(x) \) where \( p_j(x) \) is the polynomial of degree at most \( n \) that interpolates \( f \) at the \( n+1 \) consecutive points \( x_{j-k}, \ldots, x_{j-k+n} \). Show that \( S_m \) converges to \( f \) uniformly on \([0, 1]\) as \( m \to \infty \). (Hint: Use the Lagrange interpolation formula and change the variable \( x \) to \( s \) by \( x = x_{j-k} + sh \).)

(4) Let \( q_k, k = 0, 1, \ldots, n \) be a set of orthogonal polynomials on \((-1, 1)\) with weight function \( w(x) = 1 - |x| \), where \( q_k \) has degree \( k \) and leading term \( x^k \).
(a). Find \( q_0, q_1 \) and \( q_2 \).
(b). Find the Gaussian quadrature formula for
\[
\int_{-1}^{1} (1 - |x|) f(x) dx
\]
using the roots of \( q_2 \) and verify its degree of precision.
(c). Show that the Gaussian quadrature rule
\[
\int_{-1}^{1} (1 - |x|) f(x) dx \approx G_n(f) = \sum_{k=1}^{n} A_k f(x_k)
\]
has all positive coefficients \( A_k \).
(5) Two matrices $A, B \in \mathbb{C}^{n \times n}$ are unitary equivalent if $A = QBQ^*$ for some unitary matrix $Q \in \mathbb{C}^{n \times n}$. Is it true or false that $A$ and $B$ are unitary equivalent if and only if they have the same singular values? Prove or show a counterexample.

(6) Assume that the linear system
\[\begin{align*}
    r_{11}x + r_{12}y &= b_1 \\
    r_{22}y &= b_2
\end{align*}\]
where $r_{ij}$ and $b_i$ are floating point numbers is solved by back substitution using floating point arithmetic with the machine accuracy $\epsilon$. Show that the back substitution algorithm is backward stable in the sense that the computed solution $\tilde{x}$ and $\tilde{y}$ satisfy
\[\begin{align*}
    \tilde{r}_{11}\tilde{x} + \tilde{r}_{12}\tilde{y} &= b_1 \\
    \tilde{r}_{22}\tilde{y} &= b_2
\end{align*}\]
for some $\tilde{r}_{11}, \tilde{r}_{12},$ and $\tilde{r}_{22}$ that satisfy
\[|\tilde{r}_{ij} - r_{ij}|/|r_{ij}| \leq 2\epsilon + O(\epsilon^2).\]

(7) Assume that $A$ is a symmetric $n \times n$ matrix. Let $\mu$ and $x$ be an approximate eigenvalue and an approximate eigenvector respectively with $\|x\|_2 = 1$. Let $r$ be the residual in the sense that $r = Ax - \mu x$. Show that there exists an eigenvalue $\lambda$ of $A$ such that $|\mu - \lambda| \leq \|r\|_2$.

(8) Show that the Jacobi iteration converges for 2 by 2 symmetric positive definite systems.