NUMERICAL ANALYSIS QUALIFYING EXAM
Fall, 2003

(Do at least 3 problems from problems 1-4, and do at least 3 problems from problems 5-8, you may do as many as you can)

1. Let $A_n$ be a symmetric positive-definite matrix of the form

$$A_n = \begin{bmatrix} A_{n-1} & b \\ b^T & a_{nn} \end{bmatrix}.$$ 

(a). Show that if $A_{n-1}$ has a Cholesky factorization

$$A_{n-1} = L_{n-1}L_{n-1}^T,$$

with $L_{n-1}$ nonsingular, then there exist a vector $c$ and a real number $d$ such that

$$A_n = \begin{bmatrix} L_{n-1} & 0 \\ c^T & d \end{bmatrix} \begin{bmatrix} L_{n-1}^T & c \\ 0 & d \end{bmatrix}.$$ 

(b). Use the observation in part (a) in an inductive argument to prove that every symmetric positive-definite matrix has a Cholesky factorization.

2. Let $A$ be nonsingular and suppose that $A = LU = \hat{L}\hat{U}$, where $L$ and $\hat{L}$ are unit lower-triangular (i.e., their diagonal entries are all 1s) and $U$ and $\hat{U}$ are upper-triangular. Show that $L = \hat{L}$ and $U = \hat{U}$.

3. For the linear system

$$x + \alpha y = a$$
$$-\alpha x + y = b$$

(a). Write out the Jacobi method, the Gauss-Seidel method, and the SOR method for the system.

(b). Under what conditions on $\alpha$ will Jacobi and Gauss-Seidel converge?

(c). Under what conditions on $\alpha$ and $\omega$ will SOR converge?

4. Let $A$ be a real symmetric matrix whose eigenvalues $\lambda_j$, $j = 1, \ldots, m$, satisfy

$$|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_m|.$$ 

Denote by $q_j$ a unit eigenvector corresponding to $\lambda_j$. The power iteration is the following algorithm:

$v^{(0)}$ is some vector with Euclidean norm $\|v^{(0)}\| = 1$.

for $k = 1, 2, \ldots$

$w = Av^{(k-1)}$ apply $A$

$v^{(k)} = w/\|w\|$ normalize

$\lambda^{(k)} = (v^{(k)})^T A v^{(k)}$ Rayleigh quotient
Show that if $\mathbf{q}_1^T \mathbf{v}^{(0)} \neq 0$, then

$$\|\mathbf{v}^{(k)} - (\pm \mathbf{q}_1)\| = O(|\lambda_2/\lambda_1|^k),$$

and

$$|\lambda^{(k)} - \lambda_1| = O(|\lambda_2/\lambda_1|^{2k}).$$

The $\pm$ sign means that at each step $k$, one or the other choice of sign is to be taken, and then the indicated bound holds.

5. Suppose

$$\frac{dy}{dx} = f(x, y),$$

$$y(x_0) = y_0,$$

and let

$$y_1(h) = y_0 + hf(x_0, y_0)$$

$$y_2(h) = y_0 + h\frac{f(x_0, y_0) + f(x_0 + h, y_1(h))}{2}$$

Assuming $y \in C^\infty(\mathbb{R})$ and $f \in C^\infty(\mathbb{R} \times \mathbb{R})$, show that

$$|y_2(h) - y(x_0 + h)| = O(|h|^3). \quad (1)$$

6. The Chebyshev polynomials are defined by $T_n(x) = \cos(n \cos^{-1}(x))$.

(a) Show the Chebyshev polynomials are actually polynomials.

(b) Find a weight function $w(x)$ so the Chebyshev polynomials are orthogonal on $[-1, 1]$. Justify your work.

7. Approximate $\int_0^4 \frac{e^{-x}}{\sqrt{x}} \, dx$. Show your work and give a bound for the error in your approximation.

8. Given the data points $(-2, -4), (-1, -1), (0, 1), (1, 4)$, approximate $y(0.5)$ using a natural cubic spline interpolation.