1. In some computers, division \(1/a\) is done by using Newton’s method. The operation becomes finding the root of the function \(f(x) = a - 1/x\). Consider Newton’s method for this function, write down the Newton’s iteration for it (the operation only involves addition, subtraction and multiplication).

(a) Introducing the scaled residual \(r_n = 1 - ax_n\), find the relation of \(r_{n+1}\) and \(r_n\).

(b) Find the condition that guarantees the convergence.

(c) Using the results, find \(\prod_{n=0}^{\infty} (1 + r^{2^n})\), where \(|r| < 1\)

(Hint: Let \(r = r_0\), write \(x_n\) in terms of \(x_0\) and \(r_0\))

2. Suppose \(S_{na}(x), S_{cl}(x)\) are the natural cubic spline interpolant and clamped cubic spline interpolant respectively for a function \(f(x)\) with knots \(t_0, t_1, \ldots, t_n\). That is, they both are cubic spline interpolating \(f(x)\) at \(a = t_0 < t_1 < \cdots < t_n = b\), besides \(S_{na}'(t_0) = 0, S_{na}''(t_n) = 0; S_{cl}'(t_0) = f'(t_0), S_{cl}''(t_n) = f'(t_n)\). Which of the two splines has a smaller \(\int_a^b [S''(t)]^2 dt\)? Justify your answer.

3. Construct a polynomial \(f(x)\) with suitable degree such that \(f[x_0, x_1, \ldots, x_n, x] = x^r\), where \(r\) is a natural number, \(x_0 < x_1 < \cdots < x_n\) are real numbers, and \(f[x_0, x_1, \ldots, x_n, x]\) is the Newton divided difference. (Hint: consider the error formula of an interpolation polynomial in terms of the divided difference)

4. Prove the following theorem for Gaussian quadrature:

Let \(I(f) = \int_a^b f(x)w(x)dx\), where \(w(x)\) is a positive weight function, be approximated by a quadrature formula \(I_n(f) = \sum_{i=1}^n A_if(x_i)\). Then the quadrature formula \(I_n(f)\) has a maximum degree of precision of \(2n - 1\). This is attained if and only if \(x_1, x_2, \ldots, x_n\) are the zeros of \(p_n(x)\), the \(n\)th orthogonal polynomial, with the inner product \((f, g) = \int_a^b f(x)g(x)w(x)dx\).

(Hint: consider the form of \(f[x_1, x_2, \ldots, x_n, x]\))

5. Can a matrix of \(m \times n\) has a right inverse and a left inverse that are not equal? Justify your answer.

6. Given a linear system \(Ax = b\) where \(A \in C^{m \times n}, x \in C^{n \times 1}\), and \(b \in C^{m \times 1}\). The system may have no solution (inconsistent) or have a unique solution, or have non-unique solutions. The minimal solution of the system is defined as follows:

let

\[ \rho = \inf \{ \|Ax - b\|_2 : x \in C^n \} \]

Then the minimal solution is the element of least norm in the set \(K = \{ x : \|Ax - b\|_2 = \rho \}\).

Prove the theorem: The minimal solution of the equation \(Ax = b\) is given by

\[ x = A^+b \]
where $A^+$ is the pseudoinverse of $A$, defined as $A^+ = Q^*D^+P^*$, if the singular-value decomposition of $A$ is $A = PDQ$, where $P$ is an $m \times m$ unitary matrix, $D$ is an $m \times n$ diagonal matrix, $Q$ is an $n \times n$ unitary matrix, $D^+$ is an $n \times m$ diagonal matrix, $P^*$ is the Hermitian transpose of $P$ and

$$D = \begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\vdots \\
\sigma_r \\
0 \\
\vdots \\
0
\end{bmatrix}, \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0, r \leq \min(m,n),$$

$$D^+ = \begin{bmatrix}
\sigma_1^{-1} \\
\sigma_2^{-1} \\
\vdots \\
\sigma_r^{-1} \\
0 \\
\vdots \\
0
\end{bmatrix}.$$

(Hint: start from the expression of $\rho$)

7. Find explicitly (i.e., find the numerical value of every entry of) the iterative matrix in the Gauss-Seidel iterative method for solving a linear system $Ax = b$ when

$$A = \begin{bmatrix}
2 & -1 \\
-1 & 2 & -1 \\
-1 & 2 & -1 \\
& \ddots & \ddots & \ddots \\
& & -1 & 2 & -1 \\
& & & -1 & 2
\end{bmatrix}.$$

8. Consider a matrix $A$ that does not have any zero off-diagonal entry. Prove that if an eigenvalue $\lambda$ of $A$ lies on the boundary of the union of the Gershgorin Circles of the matrix $A$, then the circumference of every Gershgorin Circle passes through $\lambda$. (Hint: consider the proof of the Gershgorin theorem: consider the eigenvector corresponding to $\lambda$, and the component $x_k$ of the eigenvector which has the maximum magnitude, then consider other components)