1. (a) Assume that \( a \neq 0 \) and \( b^2 - 4ac > 0 \) and consider the equation \( ax^2 + bx + c = 0 \). The roots can be computed with the quadratic formulas
\[
x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a},
\]

Improve these formulas so that it is good even in the case \(|b| \approx \sqrt{b^2 - 4ac}\).

(b) Improve the following formula for numerical computation:
\[
\ln(1 + x) - \ln x,
\]
where \( x \) is large

2. Suppose \( f \in C^2(R) \), and \( f(p) = 0 \) implies \( f'(p) \neq 0 \).

(a) Show if \( f(p) = 0 \), then there is a \( \delta \) such that if \(|x_0 - p| < \delta\), then Newton’s method starting at \( x_0 \) converges to \( p \).

(b) Show that if \( p_1, p_2 \) are successive zeros of \( f \) (i.e. \( f(x) \neq 0 \) for \( x \in (p_1, p_2) \)) and \( p_3 \) is another zero of \( f \), then there is an \( x_0 \in (p_1, p_2) \) such that Newton’s method starting from \( x_0 \) converges to \( p_3 \). (You may just use a geometrical way to show it)

3. Suppose that the Lagrange interpolation formula for the function \( f \) at the \( n + 1 \) distinct nodes \( x_0, x_1, \ldots, x_n \) is given by
\[
P_n(x) = \sum_{j=0}^{n} l_{j,n}(x) f(x_j),
\]
where the Lagrange polynomial coefficients are given by
\[
l_{j,n}(x) = \prod_{i=0}^{n} \frac{(x - x_i)}{(x_j - x_i)}.\]

Show that for any \( n \geq 1 \),
\[
\sum_{j=0}^{n} x_j l_{j,n}(x) = x.
\]

4. Suppose a numerical formula \( I_h \) (like a numerical integration formula) using step size \( h \) is used to approximate a mathematical expression \( I \) (like a definite integral). If the error of the formula is given by
\[
I_h - I = kh^p + O(h^{p+2}), \quad \text{where } k, p \text{ are constants}
\]

(a) describe Richardson extrapolation which uses \( I_h, I_{h/2} \) to generate a more accurate numerical formula \( \tilde{I}_{h/2} \).

(b) Apply Richardson extrapolation to the trapezoidal rule
\[
I(f) = \int_a^b f(x)dx \approx I_h(f) = \frac{h}{2}(f(a) + f(b)), \quad h = b - a
\]
to derive a more accurate integration formula. Identify this more accurate integration formula by giving its familiar name. (Hint: \( I_{h/2} \) would use two subintervals)
5. Establish a finite difference formula to approximate \( \frac{\partial f(x,y)}{\partial x} \) using \( f(x,y), f(x-h,y), f(x-2h,y) \). Be as accurate as possible and derive an expression of the truncation error. Assume \( f(x,y) \) is smooth enough.

6. Give an upper bound for the relative error in the solution of the system of linear equations

\[ Ax = b \]

with symmetric matrix \( A \) given by

\[
A = \begin{bmatrix}
6 & 1 & 2 \\
1 & 6 & 2 \\
2 & 2 & 8 \\
\end{bmatrix}
\]

when the relative error in \( b \) is less that \( 4 \cdot 10^{-4} \), i.e.

\[
\frac{\| \delta b \|}{\| b \|} < 4 \cdot 10^{-4}.
\]

Use spectral norms, i.e., use

\[
\| b \| = \left( \sum_j |b_j|^2 \right)^{1/2}.
\]

7. Let \( A = (a_{ij}) \) be an \( n \times n \) matrix. An iterative scheme for the solution of the linear system \( Ax = b \) is described by

given \( x_i^{(0)} \), \( i = 1, \ldots, n \);

\[
a_{ii}y_i^{(k+1)} = b_i - \sum_{j=1}^{i-1} a_{ij}y_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}
\]

\[
x_i^{(k+1)} = \omega y_i^{(k+1)} + (1 - \omega)x_i^{(k)}, \quad i = 1, \ldots, n; \quad k = 0, 1, \ldots
\]

(a) Write the iterative scheme in the form

\[
x^{(k+1)} = Tx^{(k)} + c
\]

(Hint: consider the splitting \( A = D - L - U \)).

(b) For the particular case

\[
A = \begin{bmatrix}
2 & 1 \\
1 & 2 \\
\end{bmatrix}
\]

verify that the choice \( \omega = 8/7 \) gives the best rate of convergence.

8. Given \( x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n \) (\( T \) means transpose), define \( v = x + \text{sign}(x_1) \| x \|_2 e_1 \), where \( e_1 = (1, 0, \ldots, 0)^T \). The Householder matrix (Householder transformation) with \( v \) (Householder vector) is given by

\[
P = I - 2\frac{vv^T}{v^Tv},
\]

which is orthogonal and symmetric.

(a) Verify that \( Px = -\text{sign}(x_1) \| x \|_2 e_1 \).

(b) Describe how the Householder matrices can be used to construct an orthogonal matrix \( Q \) for a given matrix \( A \in \mathbb{R}^{n \times n} \) such that

\[
A = QR,
\]

where \( R \) is upper triangular.