1. Consider evaluating \( \cos x \) for large \( x \) by using the Taylor approximation,
\[
\cos x \approx 1 - \frac{x^2}{2!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!},
\]
If one uses it to evaluate \( \cos 2\pi = 1 \), determine \( n \) so that the Taylor approximation error is less than .0005. Suppose one does the computation using 4-digit rounding, what trouble will one encounter? How should \( \cos x \) be evaluated for large values of \( x \)?

2. Suppose \( f \in C^2(\mathbb{R}) \), and \( f(p) = 0 \) implies \( f'(p) \neq 0 \).

(1) Show if \( f(p) = 0 \), then there is a \( \delta \) such that if \(|x_0 - p| < \delta\), then Newton’s method starting at \( x_0 \) converges to \( p \).

(2) Show that if \( p_1, p_2 \) are successive zeros of \( f \) (i.e. \( f(x) \neq 0 \) for \( x \in (p_1, p_2) \)) and \( p_3 \) is another zero of \( f \), then there is an \( x_0 \in (p_1, p_2) \) such that Newton’s method starting from \( x_0 \) converges to \( p_3 \).

3. Suppose \( A \in \mathbb{R}^{n \times n} \), \( A^T \) (the transpose of \( A \)) is diagonally dominant, i.e.,
\[
|a_{ii}| \geq \sum_{i=1, i \neq j}^{n} |a_{ij}|,
\]
and \( A \) is nonsingular, show that \( A = LU \) with \( L \) being a unit lower triangular matrix, i.e., Gauss elimination can be performed without pivoting, and \(|l_{ij}| \leq 1\), where \( l_{ij} \) are entries in \( L \).

4. Suppose \( B \in \mathbb{R}^{n \times n} \) is symmetric, positive definite.

(1) Define \( \|x\| \equiv \sqrt{x^t B x}, \forall x \in \mathbb{R}^n \) (where \( x^t \) is the transpose of \( x \)). Show that this defines a norm in \( \mathbb{R}^n \) (it is called an elliptical norm).

(2) A norm in \( \mathbb{R}^n \) is monotonic if
\[
|x_i| \leq |y_i|, i = 1, 2, \ldots, n, \text{ implies } \|x\| \leq \|y\|.
\]
Construct an example to show that elliptical norms are not monotonic in general.

5. Suppose \( A \in \mathbb{R}^{m \times n} \), with \( m < n \), and \( w \in \mathbb{R}^n \). Define
\[
B = \begin{bmatrix} A \\ w^T \end{bmatrix},
\]
Show \( \sigma_1(B) \geq \sigma_1(A) \) and \( \sigma_{m+1}(B) \leq \sigma_m(A) \). Thus, the condition grows if a row is added to \( A \). (Recall that the 2-norm condition number of \( A \) is defined as \( \sigma_1(A)/\sigma_m(A) \), where \( \sigma_1(A) \) and \( \sigma_m(A) \) are the largest and smallest singular values of \( A \) respectively).

6. Suppose \( A \in \mathbb{R}^{n \times n} \), and all its off-diagonal entries are small compared to some diagonal entries. (For example, \( A \) may be a matrix obtained during the procedure of the Jacobi method) Gerschgorin theorem can be used to give a good approximate location of some eigenvalues. The Wilkinson Correction Procedure sharpens the approximation with a little more work by
multiplying the ith row of $A$ by a small number $\alpha$ and multiplying the ith column of $A$ by $\alpha^{-1}$. Suppose

\[
R_i = \left\{ z \in C : |z - a_{ii}| \leq \sum_{j=1, j \neq i}^{n} \alpha |a_{ij}| \right\}
\]

is disjoint from all the disks

\[
\left\{ z \in C : |z - a_{kk}| \leq (\alpha^{-1} |a_{ki}| + \sum_{j=1, j \neq k,i}^{n} |a_{kj}|) , \quad \forall k \neq i. \right. \]

Show that $R_i$ contains precisely one eigenvalue of $A$ (notice the approximate location of this eigenvalue has been sharpened by the procedure).

7. Find, with proof, the monic polynomial of degree of 4, $P(x)$, such that

\[\max_{-1 \leq x \leq 1} |P(x)|\]

is minimized.

8.

(1) Find the first three monic orthogonal polynomials on the interval $[0, 1]$ with respect to weight function $\ln(1/x)$.

(2) Suppose the answer to (1) are given by

\[
\psi_0(x) = 1, \psi_1(x) = x - \frac{1}{4}, \psi_2(x) = x^2 - \frac{5}{7}x + \frac{17}{252}.
\]

Derive the two-point Gaussian quadrature formula for

\[I(f) = \int_{0}^{1} f(x) \ln \left( \frac{1}{x} \right) \, dx\]

in which the weight function is $w(x) = \ln(1/x)$. What is the error of the quadrature formula (assuming that $f$ is smooth enough)?