1. Let $A$ be an $n \times n$ matrix, and let $A^\circ$ be the conjugate transpose of $A$, and $\rho(A)$ be the spectral radius of $A$. Recall that

$$\| A \|_n = \sup_{\| x \|_n = 1} \| A x \|_n \text{ and } \| x \|_n = \left\{ \sum_{i=1}^n |x_i|^n \right\}^{\frac{1}{n}}$$

Prove:
(a) $$\| A \|_1 = \max_j \sum_{i=1}^n |a_{ij}|.$$ 
(b) $$\| A \|_2^2 = \sqrt{\rho(A^\circ A)}.$$ 

2. Let $A$ be a strictly diagonally dominant matrix. Prove the Jacobi iteration method converges for $A$ independently of the choice of initial vector.

3. Consider the application of the iteration

$$z_{k+1} = \frac{z_k^2 + 2}{3}$$

to the equation $x^2 - 3x + 2 = 0$.

(a) Does the method converge? Why?
(b) Show that $x_k \to 1$ as $k \to \infty$ if $-2 < z_0 < 2$. (First prove that $z_{k+1}$ is between $z_k$ and 1 when $k \geq 1$.)
(c) Show that $z_{k+1} = 2$ if $z_0 = \pm 2$, $k = 0, 1, \ldots$, but that the convergence to the root $\alpha = 2$ for any other value of $z_0$ is impossible.

4. (a) Define a Householder transformation.
(b) Show that for any square matrix $A$, there is a unitary matrix $U$ such that $U^0A U$ is upper triangular. ($U^0$ is the conjugate transpose of $U$).

5. Suppose $A = (a_{ij})$ is an $n \times n$ matrix and that

$$|a_{ij}| \geq C \sum_{j \neq i} |a_{ij}|$$

for $i = 1, \ldots, n$ where $C > 1$. Let $K(A) = \rho(A) \rho(A^{-1})$ where $\rho(A)$ is the spectral radius of $A$. Find a bound for $K(A)$ in terms of $C$ and the $a_{ij}$'s.

6. State the Weierstrase approximation theorem, define Bernstein polynomials, and explain how the latter are used in the proof of the former.

7. Assume

$$f(0) = 1, f(1) = 3, f(2), \text{ and } f(3) = 1.$$
(a) Write a formula for the polynomial, \( P(x) \) of degree at most 3 that interpolates \( f \) at these points.

(b) Derive an expression for the error \( f(x) - P(x) \) at an arbitrary point \( x \). Be sure to indicate the conditions assumed on \( f \).

8. Prove: Of all \( n \)th degree monic polynomials \( P_n(x) \), The Chebyshev polynomial

\[
T_n(x) = \frac{1}{2^n} \cos(n \cos^{-1} x)
\]

has the smallest maximum norm on the interval \([-1, 1]\), that is,

\[
\max_{-1 \leq x \leq 1} |T_n(x)| \leq \max_{-1 \leq x \leq 1} |P_n(x)|.
\]

9. Assume that \( f(x) \) has a continuous fourth derivative on an open interval containing the interval \([c - h, c + h]\). Derive an expression for the error in approximating \( f''(c) \) by

\[
\frac{f(c + h) + f(c - h) - 2f(c)}{h^2}.
\]

10. Use undetermined coefficients to find coefficients \( H_1, H_2, \) and \( H_3 \) so that the approximation

\[
\int_c^b f(x)dx \approx H_1 f\left(\frac{2a + b}{3}\right) + H_2 f\left(\frac{a + b}{2}\right) + H_3 f\left(\frac{a + 3b}{4}\right)
\]

is exact for polynomials of degree less than or equal to 2.

11. Show that the numerical integration method

\[
y_{n+1} = y_{n-3} + \frac{4h}{3} \left(2y_n' - y_{n-1}' + 2y_{n-2}'\right)
\]

to approximating the solution to the ordinary differential equation \( y' = f(x, y), y_0 = y(x_0) \) is exact for polynomials of degree 0 or 1. Is the method stable? Explain.