1. Prove the following theorem: If \( f \in C^2(a,b), f'(x)f''(x) \neq 0 \), and \( f(x) \) has a zero in \((a,b)\), then the zero is unique in \((a,b)\), and the Newton iteration will converge to it if the starting value \( x_0 \) and the first approximation \( x_1 \) are both in \((a,b)\). (You may just do a special case where \( f'(x) < 0, f''(x) < 0 \) in \((a,b)\)).

2. Suppose a numerical integration formula \( I_n \) using \( n \) subintervals to approximate the definite integral \( I = \int_a^b f(x) \, dx \) has an error given by \( I - I_n = cn^p \) where \( c, p \) are constants. Derive the computable estimate

\[
\frac{I_{2n} - I_n}{I_{4n} - I_{2n}} \approx 2^p
\]

This gives a practical means of checking the value of \( p \), using three successive values \( I_n, I_{2n}, \) and \( I_{4n} \).

3. By considering the proof of

\[
f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1)\ldots(x - x_n),
\]

where \( p_n(x) \) is the polynomial of degree less than or equal to \( n \), which interpolates \( f(x) \) at \( n + 1 \) nodes \( x_0, x_1, \ldots, x_n \). Find that the error formula for

\[
f(x) - p_m(x),
\]

where \( p_m(x) \) is a polynomial of degree greater than \( n \), which interpolates \( f(x) \) at \( n + 1 \) nodes \( x_0, x_1, \ldots, x_n \).

4. Prove the following theorem: Define a set of functions

\[
P^n_M = \{ p \in P^n \mid \max_{x \in [a,b]} |p(x)| \leq M \}
\]

where \( P^n \) is the linear space of the polynomials of degree less than or equal to \( n \). Then there is a constant \( C > 0 \) such that for every \( p \in P^n_M \) and \( x \in [a,b] \) and any positive integer \( k \),

\[
\left| \frac{d^k p(x)}{dx^k} \right| \leq C.
\]

(Hint: Chebyshev polynomials of degree 0, 1, \ldots, \( n \) form a basis for \( P^n \))

5. A matrix norm is defined as

\[
\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|.
\]

Prove or disprove: \( \|AB\|_\infty = \|A\|_\infty \|B\|_\infty \). What about the special case: \( \|A^2\|_\infty = \|A\|_\infty \|A\|_\infty \)?
6. Find the explicit form for the iterative matrix in the Gauss-Seidel iterative method for solving a linear system \( Ax = b \) when

\[
A = \begin{bmatrix}
2 & -1 & & \\
-1 & 2 & -1 & \\
& -1 & 2 & -1 \\
& & -1 & \\
& & & \ddots & -1 \\
& & & & \ddots & \ddots & \\
& & & & & -1 & 2 & -1 \\
& & & & & & -1 & 2
\end{bmatrix}.
\]

7. Suppose \( A \) is an invertible matrix and that \( B \) is a matrix with \( \| B - A^{-1} \| \leq \delta \| A^{-1} \| \). Let \( \{x_n\}_{n=0}^\infty \) be the sequence of vectors generated by the algorithm

(i) \( r_n = b - Ax_n \)

(ii) \( x_{n+1} = x_n + Br_n \)

with a given starting value \( x_0 \). Give a sufficient condition on the size of \( \delta \) for the sequence to converge to the solution of the linear system \( Ax = b \) for arbitrary starting value \( x_0 \). Prove that your condition is correct.

8. Describe an algorithm that reduces a square real matrix to a lower Hessenberg matrix without changing its eigenvalues. (A matrix \( A \) is lower Hessenberg if \( a_{ij} = 0 \) provided \( j - i > 1 \).)