1. Let \( \pi : S^2 \to \mathbb{RP}^2 \) be the standard covering projection. Prove that there is no map \( f : \mathbb{RP}^2 \to S^2 \) so that \( \pi \circ f = \text{id} \).

2. Recall that

\[
d\alpha(X_0, \ldots, X_p) = \sum_{k=0}^{P} (-1)^k X_k \alpha(X_0, \ldots, \hat{X}_k, \ldots X_p)
+ \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], \ldots, \hat{X}_i, \ldots \hat{X}_j \ldots X_p)
\]

Prove that \( d\alpha(X_0, \ldots, X_p) = \sum_{k=0}^{P} (-1)^k (\nabla X_k \alpha)(X_0, \ldots, \hat{X}_k, \ldots X_p) \).

3. (a) Give the definition of a Lie group.
   (b) Give the definition of a Lie algebra.
   (c) Give the definition of a representation of a Lie group, \( \mu : G \to \text{Aut}(V) \).
   (d) Give the definition of a representation of a Lie algebra, \( \dot{\mu} : g \to \text{End}(V) \).
   (e) Define the Lie algebra of a Lie group.
   (f) Describe how a representation of a Lie group induces a representation of the corresponding Lie algebra and prove that the induced representation is a Lie algebra representation.

4. Prove that the holonomy of a simply connected Riemannian manifold is connected.
5. Let \( X = \frac{\partial}{\partial x} \) and \( Y = \frac{\partial}{\partial x} + (x^2 + 1) \frac{\partial}{\partial y} \) on \( \mathbb{R}^2 \).

(a) Compute \([X, Y]\).

(b) Compute the flow of \( X \).

(c) Compute the flow of \( Y \).

(d) Let \( F^Z : \mathbb{R} \times M \to M \) be the flow of a vector field \( Z \). If \( F^Z_s \circ F^W_t = F^W_t \circ F^Z_s \) for all \( s \) and \( t \), what can you say about \([Z, W]\)? Why?

(e) Is there a function \( f_Y : \mathbb{R}^2 \to \mathbb{R} \) so that \( F^{f_X}_s \circ F^Y_s = F^Y_s \circ F^{f_X}_s \) for all \( s \) and \( t \)? Why?

6. Let \( f : \mathbb{R}^3 \to \mathbb{R} : f(x, y, z) = xy - z \).

\( \Sigma = f^{-1}(0) \cap \{(x, y, z)|x^2 + y^2 \leq 1\} \)

(a) Verify that \( \Sigma \) is a manifold.

(b) Compare the orientation induced on \( \Sigma \) using \( \nabla f / |\nabla f| \) and \( dx \wedge dy \wedge dz \) with the orientation \( dx \wedge dy \).

(c) Compute \( \int_{\Sigma} \frac{|\nabla f \circ \kappa|}{|\nabla f|} \, dx \wedge dy \) when \( \Sigma \) is oriented by \( dx \wedge dy \). What does this represent?

7. The connected sum \( M_1 \# M_2 \) of two oriented \( n \)-manifolds \( M_1, M_2 \) is defined as \((M_1 \setminus \text{int} B^n) \cup_{S^{n-1}} (M_2 \setminus \text{int} B^n)\), where \( B^n \) is a ball in \( M_1(M_2) \) and \( S^{n-1} \) is its boundary.

(a) Show that if \( n \geq 3 \), then \( \pi_1(M_1 \# M_2) = \pi_1(M_1) \ast \pi_1(M_2) \).

(b) Compute the fundamental group of \( T^2 \# T^2 \) (where \( T^2 \) is the 2-dimensional torus).

[Hint: What is \( \pi_1(T^2 \setminus \text{int} D^2) \)?]

8. (a) Show that there exists a natural map \( S^1 \times S^3 \to U(2) \) with discrete fiber by using the Lie group structure of \( S^1 \) and \( S^3 \).

(b) What is the fiber?

(c) Using the result above, what is \( \pi_1 U(2) \)?