Geometry of Manifolds Qualifying Exam
Fall 1997

Part A. Do all nine (9) questions in part A.

1. What is the fundamental group of
   (a) \( \mathbb{RP}^2 \) (the real projective plane)
   (b) \( S^1 \times S^1 \)
   (c) \( T(M) \), the total space of the tangent bundle to a simply connected smooth manifold, \( M \).

2. Describe in detain the flows of the vectorfield on \( \mathbb{R}^2 \) given by
   \[ -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \]

3. Let \( \omega \) be the 1-form on \( \mathbb{R}^2 \) given by \( x(x-1)(y-1)dx \), and let \( R \) be the region
   \[ \{(x,y)|0 \leq x \leq 1, 0 \leq y \leq 1\} \]
   Find \( \int_R d\omega \).

4. (a) Give an example of a compact orientable manifold with non-trivial tangent bundle.
   (b) Give an example of a compact orientable manifold with trivial tangent bundle.
   (c) Give an example of a compact non-orientable manifold.

5. If \( M = S^1 \times S^4 \), what is the dimension of the fibres of the third exterior bundle \( \Lambda^3(M) \)?

6. How many non-zero vectorspaces of differential forms are there in the deRham complex of \( S^2 \times S^2 \)?

7. What is the scalar curvature of the surface \( 3x + 2y - z = 0 \) in \( \mathbb{R}^3 \) at the point \((0,0,0)\)?

8. Give an example of a locally Euclidean topological space which is not a topological manifold.

9. State the deRham Theorem.

Part B. Choose four (4) and only four of the following problems.

1. On \( \mathbb{R}^3 \) with standard Euclidean coordinates \((x, y, z)\), consider the 2-form \( \alpha = f(x, y, z)dx \wedge dy + yzdx \wedge dz + x^2dy \wedge dz \). Choose a function \( f(x, y, z) \) so that \( d\alpha = 0 \) and \( \alpha|_{z=1} = dx \wedge dy \).

2. (a) Define the deRham cohomology groups of a differentiable manifold.
   (b) Calculate the deRham cohomology groups of the circle \( S^1 \) directly from the definition in part (a).

3. Give a detailed computation of the fundamental group of the closed compact surface of genus 2 (a.k.a the “two-holed torus”).

4. (a) Write down the deRham cohomology groups for the 4-sphere \( S^4 \).
   (b) Suppose that \( \omega \) is a differential 2-form on \( S^4 \) and that \( d\omega = 0 \). Show that
   i. \( \omega \wedge \omega = d\phi \) for some 3-form \( \phi \).
ii. $\int_{S^4} \omega \wedge \omega = 0$.

iii. There is at least one point $x \in S^4$ such that $\omega \wedge \omega(x) = 0$.

5. (a) Define what we mean by a Lie group.

(b) If $G$ is a Lie group, define its Lie algebra $g$.

(c) Apply the construction of b) to determine the Lie algebra of $SO(3)$, including a derivation of the bracket.

(d) Show that the tangent bundle to a Lie group is equivalent to a trivial (product) bundle.

6. Let $(M,g)$ be a Riemannian manifold and $V(M)$ be the smooth vectorfields over $M$.

(a) For $X,Y \in V(M)$ define the Riemannian curvature operator $R(X,Y) : V(M) \to V(M)$.

(b) Show that if $M = \mathbb{R}^n$ and $g$ is the Euclidean metric, then $R(X,Y)Z = 0$ for all vectorfields $X,Y,Z$.

(c) Suppose that $R(X,Y)Z = 0$ for all vectorfields $X,Y,Z$ on an arbitrary Riemannian manifold $(M,g)$. Sketch a proof that shows that for $x \in M$ there is a coordinate system $(x_1, \ldots, x_n)$ around $x$ such that

$$g = \sum_{i=1}^{n} dx^i \otimes dx^i$$