1. Let \( g_n : B_R(0) \to \mathbb{R} \), where \( B_R(0) \subset \mathbb{R}^n \) is the ball of radius \( R \) centered at \( 0 \), be a sequence of harmonic functions satisfying that \( g_n(x_0) \) converges for some \( x_0 \in B_R(0) \). Show that \( g_n \) converges to a harmonic function in \( B_R(0) \).

(Hint: Recall that any function that is harmonic in \( D'(\Omega) \) is harmonic in \( \Omega \), combine this with Harnack’s inequality.)

2. Show that under suitable assumptions on the function \( f : \mathbb{R} \to \mathbb{R} \), the bounded solution to

\[
(P1) \begin{cases}
  u_t = u_{xx}, & \text{for } x > 0, \ t > 0 \\
  u_x(0, t) = 0, & t > 0 \\
  u(x, 0) = f(x), & 0 < x,
\end{cases}
\]

Is given by

\[
u(x, t) = \int_0^\infty G(x, \xi, t)f(\xi) \ d\xi,
\]

where \( G(x, \xi, t) = K(x - \xi, t) - K(x + \xi, t) \) and \( K \) is the fundamental solution of the heat equation.

(Hint: extend \( f \) in a suitable way to \( (\mathbb{R} \times \mathbb{R}) \) and solve the initial value problem for the extended \( f \).)

State a sufficient condition on the growth of \( f \) for existence and uniqueness of solution to \( (P1) \) to hold.

3. Show that the problem

\[
(P2) \begin{cases}
  u'(t) = f(u(t)), \\
  u(0) = 0
\end{cases}
\]

need not be uniquely solvable in any neighborhood of \( t = 0 \) if \( f \) is continuous but not Lipschitz.

4. Suppose \( u, v \) are two solutions of

\[
\frac{d^2f}{dt^2} + 3t \frac{df}{dt} - f = 0
\]

with \( W(u, v) \neq 0 \) for all \( t \) where \( W \) denotes the Wronskian. Show that the zeros of \( u \) and \( v \) interlace, i.e. if \( v(p) = 0 = v(q) \), \( p < q \), then there exists an \( r, p \leq r \leq q \) with \( u(r) = 0 \) and similarly there is a zero of \( v \) between every pair of zeros of \( u \).
5. Let $J_0(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(t \cos(\theta)) \, d\theta$. Show that $J_0(t)$ solves the initial value problem

$$ty'' + y' + ty = 0, \quad y(0) = 1.$$ 

(Hint: write everything in terms of power series.)

6. Suppose $u$ solves

$$u_{xx} + 3u_{xt} + 2u_{tt} = 0$$
$$u(x, 0) = F(x)$$
$$u_t(x, 0) = G(x)$$

where $F, G \in C^2(\mathbb{R})$. Suppose also $F(x) = 0 = G(x)$ for $-1 \leq x \leq 1$. For what region of $\{(x, t) : t > 0\}$ can you conclude $u(x, t) = 0$?