1) Solve the Cauchy problem
\[ \cos(y) \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + 2 \tan(y)u - 2 \tan(y) = 0, \quad u(x, 0) = h(x) \]

2) a) Find a 2π-periodic in x solution of the equation
\[ u_t - u_{xx} = 0 \]
that satisfies the initial condition
\[ u(0, x) = \phi_N(x), \]
where \( N \geq 1 \) is an integer and the function \( \phi_N \) is defined on \([-\pi, \pi]\) as follows
\[ \phi_N(x) = \begin{cases} N, & \text{if } -1/2N < x < 1/2N \\ 0, & \text{otherwise,} \end{cases} \]
and extended periodically to the whole line. How many solutions are there? In what sense?

b) Let \( u_N(t, x) \) denote the solution of the above problem such that \( u_N(t, \cdot) \) is a strongly continuous function of \( t \in [0, T] \) with values in \( L^2([-\pi, \pi]) \). Prove that such solution does exist and is unique. Is \( u_N(t, x) \) a bounded function of \( x \) for every \( t \geq 0 \)?

c) In what sense does there exist the limit \( u_\infty = \lim_{N \to \infty} u_N \)? Find \( u_\infty \).
3) State and prove the maximum principle for the equation

\[ u_t - \Delta u + e^u = 0 \]

in \( \mathbb{R}^n \).

Use the maximum principle to state and prove a uniqueness theorem.

4) Determine the type of the equation

\[ \sum_{i,j=1}^{4} A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{4} \frac{\partial u}{\partial x_i} - u = 0, \]

where \( A_{ij} \) is the following matrix:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]

State and prove the finite domain of dependence property for this equation.

5) For the equation \(-\Delta u + u = f\) on the torus \( \mathbb{T}^n \), prove the estimate

\[
\sum_{j,k=1}^{n} \|u_{x_j x_k}\|_{L^2(\mathbb{T}^n)} \leq C(\|f\|_{L^2(\mathbb{T}^n)} + \|u\|_{L^2(\mathbb{T}^n)})
\]

with a constant \( C \) independent of \( f \).

6) Find all distributional solutions of the equation

\[ xy'(x) = 1. \]

7) State and prove a finite dimensional version of Fredholm’s Alternative.

8) Consider the Cauchy problem for the wave equation on \( \mathbb{R}^n \):

\[ u_{tt} - \Delta u = 0, \quad u(0, x) = f(x), \quad u_t(0, x) = 0. \]

In the 1-dimensional case \( (n = 1) \) show that there exists a constant \( C > 0 \) such that

\[ \sup_t \|u(t, \cdot)\|_{L^1(\mathbb{R})} \leq C \|f\|_{L^1(\mathbb{R})}, \]

for all \( f \in L^1(\mathbb{R}). \)

In the 3-dimensional case \( (n = 3) \) prove that there is no constant such that the above inequality holds. (Hint: consider spherically-symmetric solutions, use the change of variables \( u(t, r) = r^\alpha v(t, r) \) with an appropriate \( \alpha \) to obtain the 1-D wave equation for \( v \), and choose \( v(0, r) \) to have a support in an annulus.)