Partial Differential Equations
Qualifying Examination
September 14, 1987

Do any 7 problems.

1. (a) What does it mean to say that a problem in partial differential equations is well-posed?
   (b) Given a region \( \Omega \) in \( \mathbb{R}^n \) and a second order linear PDE defined in \( \Omega \), what does it mean to say the PDE is elliptic in \( \Omega \)?
   (c) Consider the Cauchy problem
   \[
   u_t = \Delta u, \quad x \in \mathbb{R}^n, \quad t > 0
   \]
   \[
   u(0, x) = f(x), \quad x \in \mathbb{R}^n.
   \]

   State the Maximum Principle for the solution of this problem (be sure to state clearly all relevant hypotheses).

2. Let \( \mathcal{D} = \{ u | u \in C^2(\overline{\Omega}), u(x) = f(x) \text{ for } x \in \partial\Omega \} \) where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with smooth boundary. For \( u \in \mathcal{D} \) let
   \[
   J(u) = \int_{\Omega} |\nabla u|^2 \, dx.
   \]

   Suppose that \( u \in \mathcal{D} \) satisfies the Dirichlet problem
   \[
   \Delta u = 0 \quad \text{in } \Omega
   
   u(x) = f(x) \quad \text{on } \partial\Omega.
   \]

   Prove that \( u \) minimizes \( J \).

3. (a) Consider the initial value problem
   \[
   u_t + uu_x = 0 \quad -\infty < x < \infty, \quad t > 0
   \]
   \[
   u(x, 0) = f(x), \quad -\infty < x < \infty.
   \]

   Assume \( f \) is \( C^1 \). Show that unless \( f \) is nondecreasing on \( (-\infty, \infty) \) there cannot be a \( C^1 \) function \( u(x, t), \quad -\infty < x < \infty, \quad t \geq 0 \) which is a solution of the IVP everywhere in \( -\infty < x < \infty, \quad t \geq 0 \).
(b) Show that the IVP

\[ u_t + uu_x = 0 \quad -\infty < x < \infty, \quad t > 0 \]
\[ u(x,0) = 2x + 1 \quad -\infty < x < \infty \]

has a smooth solution by finding an explicit formula for the solution.

4. Consider the quasilinear system

\[ u_t + uu_x + \frac{c^2}{\rho} = 0 \]
\[ \rho_t + u\rho_x + \rho u_x = 0 \]

where \( u \) and \( \rho \) are unknown functions of \( x \) and \( t \) and \( c \) is a known function of \( \rho \).

(a) Show that this is a hyperbolic system provided we assume \( c(\rho) > 0 \).

(b) Find the differential equations of the characteristic curves for this system.

5. Let \( \Omega \) be the first quadrant in \( \mathbb{R}^2 \). Define \( f(x,y) \) for \( (x,y) \in \partial \Omega \) by:

\[ f(0,y) = \begin{cases} 
1 & 0 \leq y < 1 \\
0 & 1 \leq y 
\end{cases} \]
\[ f(x,0) = \begin{cases} 
1 & 0 \leq x < 1 \\
0 & 1 \leq x 
\end{cases} \]

Use complex variable methods to solve the Dirichlet problem on \( \Omega \) with boundary data \( f \).

6. Explain what is meant by "Huygens' Principle". For which dimensions does this principle hold?

7. Suppose the initial temperature in a spherical body of radius \( a > 0 \) is constant at \( U_0 \). For \( t > 0 \) the boundary is kept at temperature 0. Assume heat conduction is governed by

\[ u_t = \Delta u \]

where \( u \) is the temperature function. Derive a series representation for the solution of this problem.

8. (a) State the mean value property of harmonic functions in \( \mathbb{R}^n \).

(b) Using the result in (a), state and prove the Maximum Principle for harmonic functions on a bounded domain in \( \mathbb{R}^n \).
(c) Use the Maximum Principle to show that the solution of the Dirichlet problem on a bounded domain $\Omega$ in $\mathbb{R}^n$ is unique (if it exists).

(d) Prove that the solution of the Dirichlet problem on a bounded domain $\Omega$ in $\mathbb{R}^n$ depends continuously on the boundary condition. Include a careful statement of what this means.

9. (a) Derive the Green function for Dirichlet's problem for the Laplace equation on the upper half-plane in $\mathbb{R}^2$.
   (b) Use (a) to derive the formula
   \[
   u(x, y) = \frac{x}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)}{(x - \xi)^2 + y^2} d\xi
   \]
   for the solution of the Dirichlet problem
   \[
   \nabla^2 u = 0 \quad -\infty < x < \infty, \quad 0 < y < \infty
   \]
   \[
   u(x, 0) = f(x),
   \]
   where $f$ is continuous on $-\infty < x < \infty$.

10. Prove uniqueness of solutions for the problem
   \[
   u_{tt} = \alpha^2 \Delta u + f(t, x) \quad \text{for} \quad x \in \Omega, \quad t > 0
   \]
   \[
   u(0, x) = \phi(x), \quad u_t(0, x) = \psi(x) \quad \text{for} \quad x \in \Omega
   \]
   \[
   \tau \frac{\partial u}{\partial n} + \sigma u = 0 \quad \text{on} \quad \partial \Omega,
   \]
   where $\Omega$ is a region with smooth boundary in $\mathbb{R}^n$, $\phi$ and $\psi$ are $C^1$ on $\Omega$, and $\sigma$ and $\tau$ are positive constants. (Hint: Use the energy integral
   \[
   E(t) = \int_{\Omega} \left( \tau u_t^2 + \alpha^2 \nabla u^2 \right) dx + \int_{\partial \Omega} \alpha^2 \sigma^2 u^2 ds.
   \]

11. Use Fourier transforms to solve the Cauchy problem for the 1-dimensional heat equation with source term $f(x, t)$,
   \[
   u_t = u_{xx} + f(x, t)
   \]
   \[
   u(x, 0) = \phi(x).
   \]