COMPLEX VARIABLES QUALIFYING EXAMINATION - Spring 1998
(Bennett and Burckel)

Let $\mathbb{R}$ denote the real line, $\mathbb{C}$ the complex plane, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, $\Omega$ a non-void, open, connected subset of $\mathbb{C}$, $C(\Omega)$ the continuous $\mathbb{C}$-valued functions on $\Omega$, $H(\Omega)$ the complex-differentiable function on $\Omega$.

1. Let $S$ be the open square $]0,1[\times]0,1[$ and identify $(x,y) \in \mathbb{R}^2$ with $x + iy \in \mathbb{C}$.

(i) What does it mean for a function $f : S \rightarrow \mathbb{R}^2$ to be $\mathbb{R}$-differentiable at $(x_0, y_0) \in S$?

(ii) If $f$ is $\mathbb{R}$-differentiable at $(x_0, y_0)$, what property of its $\mathbb{R}$-derivative will make $f$ also $\mathbb{C}$-differentiable at $x_0 + iy_0$?

2. Suppose $\Omega$ is starlike with respect to its point $a$. Show that for every $f \in H(\Omega)$ the companion function $F$ defined by

$$F(z) := \int_{[a,z]} f \quad \forall z \in \Omega$$

is also holomorphic in $\Omega$ and satisfies $F' = f$.

3. What is the topology of local uniform convergence in $C(\Omega)$? Is this a metric topology? Show that:

(i) $C(\Omega)$ is complete in this topology.

(ii) $H(\Omega)$ is a closed subset of $C(\Omega)$.

Hint: For (ii) Morera’s theorem is useful.

4. Prove that $f \mapsto f'$ is a continuous mapping of $H(\Omega)$ into itself (in the topology of Problem 3). Give an example of an $\Omega$ for which this map is not surjective.

Hint: For the continuity, exploit Cauchy’s integral formula.
5. Show that if \( f \in H(\Omega) \) is one-to-one, then \( f' \) is zero-free in \( \Omega \). Is the converse true?

6. \( h : \mathbb{C} \to \mathbb{R} \) is harmonic and not constant. Prove that \( h \) has a zero.
   
   \textit{Hint:} If \( h > 0 \) throughout \( \mathbb{C} \), employ Harnack's inequalities.

7. Compute \( \int_{\Gamma} \frac{z^2 + 1}{z^2 - 1} \, dz \), where \( \Gamma \) is the indicated path.

8. The \textit{cross-ratio} of an ordered quadruple of distinct complex numbers is \( [z_1, z_2, z_3, z_4] := \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)} \). Show that \( [z_1, z_2, z_3, z_4] = [w_1, w_2, w_3, w_4] \) if and only if there is a Möbius transformation (i.e., a linear fractional transformation) that maps each \( z_j \) to \( w_j \).

9. Suppose \( \sum_{n=0}^{\infty} c_n z^n \) has radius of convergence 1. Show that the function \( f(z) := \sum_{n=0}^{\infty} c_n z^n \) which it defines in \( \mathbb{D} \) is holomorphic. Can you find such an \( f \) which can be continuously extended to \( \overline{\mathbb{D}} \)? Disprove or give an example.

10. Prove that the zeros of a non-constant polynomial depend continuously on its coefficients in the following sense: Given \( P(z) = c_0 + c_1 z + \ldots + c_n z^n \) (\( n > 0, c_n \neq 0 \)) whose (distinct) zeros are \( z_1, \ldots, z_r \) and given \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that whenever complex numbers satisfy \( |b_j - c_j| < \delta \) for all \( j \), the polynomial \( Q(z) := b_0 + b_1 z + \ldots + b_n z^n \) will have at least one zero in each of the disks \( D(z_j, \varepsilon) := \{ z \in \mathbb{C} : |z - z_j| < \varepsilon \} \) and all its zeros in the union \( \bigcup_{j=1}^{r} D(z_j, \varepsilon) \) of these disks.