R is the set of all real numbers, C the set of all complex numbers, \( D(a,r) := \{ z \in \mathbb{C} : |z - a| < r \} \) for any \( a \in \mathbb{C} \), \( 0 < r \in \mathbb{R} \), \( \mathbb{D} := D(0,1) \).

1. \( f : \mathbb{D} \to \mathbb{D} \) is holomorphic. Show that
\[
\frac{|f(x) - f(w)|}{1 - f(z)f(w)} \leq \frac{|z - w|}{1 - \overline{z}w} \quad \forall z, w \in \mathbb{D}.
\]

What can you say about \( f \) if there exists a pair \((z,w)\) for which \( z \neq w \) and equality holds above?

2. Let \( f \) be a one-to-one holomorphic function on a region \( \Omega \). Show that \( f' \) is zero-free.

3. (a) \( f \) is continuous on \( \overline{\mathbb{D}} \), holomorphic in \( \mathbb{D} \). Show that \( f \) is uniformly approximable on \( \overline{\mathbb{D}} \) by polynomials.

(b) State and prove the converse of (a).

4. Let \( A \) denote the algebra of continuous functions on \( \overline{\mathbb{D}} \) which are holomorphic in \( \mathbb{D} \). Find all \( \mathbb{C} \)-algebra homomorphisms \( \Phi : A \to \mathbb{C} \) which are not identically 0. **HINT:** Show first that \( \Phi(f) \in f(\overline{\mathbb{D}}) \) for every \( f \), then see what \( \Phi \) does to polynomials. Is \( \Phi \) continuous?

5. (a) State the Maximum Modulus Principle for holomorphic functions, and

(b) Give two different proofs of this principle.

6. Suppose that \( f \) is holomorphic on \( \overline{\mathbb{D}} \), \( |f(z)| < 1 \) whenever \( |z| = 1 \), and that \( \alpha \in \mathbb{D} \). Find the number of solutions in \( \mathbb{D} \) of the equation
\[
f(z) = \left( \frac{z - \alpha}{\alpha z - 1} \right)^2.
\]

7. Suppose \( f \) is holomorphic in the disc \( D(a,R) \). Prove that
\[
|f(a)| \leq \frac{1}{\pi R^2} \int \int_{D(a,R)} |f(x + iy)|dxdy.
\]

Is this inequality valid for harmonic functions? **HINT:** Polar coordinates.

8. (a) State and prove Harnack’s Inequalities.

(b) Using (a) [whether or not you proved it] demonstrate the following: If \( h : \mathbb{C} \to \mathbb{R} \) is harmonic but **not** constant, and

\[
m(r) := \min_{\theta} h(re^{i\theta}), \quad M(r) := \max_{\theta} h(re^{i\theta}),
\]

then
\[
\lim_{r \to \infty} \frac{m(r)}{r} < 0 < \lim_{r \to \infty} \frac{M(r)}{r}.
\]

(c) For the \( h \) in (b), infer from (b) [whether you proved (b) or not] that \( h \) must have at least one zero, then go on to prove that \( h(\mathbb{C}) = \mathbb{R} \).