In what follows \( \mathbb{R} \) is the real numbers, \( \mathbb{C} \) the complex numbers, \( \mathbb{D} \) the open unit disc centered at 0 and \( \mathbb{T} \) the boundary of \( \mathbb{D} \). The set of functions holomorphic in \( \Omega \) is denoted \( H(\Omega) \).

1. Let \( f \) be a continuous function on \( \mathbb{D} \) which satisfies \( \int_{\Delta} f = 0 \) for each triangle \( \Delta \subset \mathbb{D} \) such that one side on \( \Delta \) lies on \( \mathbb{R} \) and another side of \( \Delta \) is parallel to \( i\mathbb{R} \). Show that \( f \) is holomorphic.

2. Prove that \( 1/z \) is not uniformly approximable on \( \mathbb{T} \) by polynomials in \( z \).

3. Let \( \Omega \) be a region in \( \mathbb{C} \), \( f \in H(\Omega) \setminus \{0\} \), \( n \) a positive integer. Suppose that \( |z|^n f(z) \) attains a maximum over \( \Omega \) at some point of \( \Omega \). Show that \( 0 \in \Omega \).

4. Prove that for all \( z \) in the open right half-plane \( \mathbb{H} \) the integral \( \int_1^\infty e^{-t} t^{z-1} dt \) exists and defines a holomorphic function of \( z \in \mathbb{H} \).

5. (i) If \( f \) is holomorphic in a neighborhood of \( \mathbb{D} \), then
   \[
   |f(z)| \leq \frac{1}{\sqrt{1 - |z|^2}} \left[ \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta \right]^{1/2} \forall z \in \mathbb{D}.
   
   (ii) Use (i) to draw the same conclusion in case \( f \) is only continuous on \( \mathbb{D} \), holomorphic in \( \mathbb{D} \).

6. \( f \) and \( g \) are each holomorphic in a neighborhood of 0, \( f(0) = 0 \) with multiplicity \( m \), \( g(0) = 0 \) with multiplicity \( n \). What is the multiplicity of 0 as a zero of \( f \circ g \)?

7. (i) Use the function \( f(t) := e^{it}, t \in [0,2\pi] \), to show that the Mean-Value Theorem of differential calculus fails (generally) for complex-valued functions.
   
   (ii) Prove that, in spite of (i), if \( F \) is holomorphic in a convex region \( \Omega \) and \( |F'| \leq M \), then
   \[
   |F(z_2) - F(z_1)| \leq M |z_2 - z_1| \forall z_1, z_2 \in \Omega.
   
8. Let \( \Omega \) be a bounded region in \( \mathbb{C} \), \( f : \overline{\Omega} \to \mathbb{C} \) a continuous non-constant function which is holomorphic in \( \Omega \) and maps \( \partial \Omega \) into \( \mathbb{T} \).
   
   (i) Show that \( 0 \in f(\Omega) \).
   
   (ii) Show that \( f(\Omega) = \mathbb{D} \).
   
   **Hint:** To get “\( \supset \)”, apply (i) to \( \phi \circ f \) for certain holomorphic maps \( \phi \) of \( \mathbb{D} \) into \( \mathbb{D} \).

9. \( f \) is continuous on \( \overline{\mathbb{D}} \), holomorphic in \( \mathbb{D} \) and \( \text{diam} f(\mathbb{T}) \leq 1 \). Show that \( \text{diam} f(r \mathbb{T}) \leq r \) for each \( 0 \leq r \leq 1 \).
   
   **Hint:** \( \text{diam} f(r \mathbb{T}) := \max \{|f(ru_1) - f(ru_2)| : u_1, u_2 \in \mathbb{T} \} \). If this is achieved at \( u_1, u_2 \) consider the holomorphic function \( F(z) := f(zu_1) - f(zu_2) \).

10. \( h : \mathbb{C} \to \mathbb{R} \) is harmonic and non-constant.
   
   (i) Prove that \( h \) is not bounded above.
   
   (ii) Prove that \( h \) is not bounded below.
   
   (iii) Prove that \( h(\mathbb{C}) = \mathbb{R} \).