1. Let $U$ be open $\subset \mathbb{C}$, $a \in U$, $f$ holomorphic in $U\{a\}$.

   (i) What does it mean to say $f$ has a pole at $a$?

   (ii) Show that if $a$ is a pole, then $f(U\{a\})$ is a “neighborhood of $\infty$", that is, its complement in $\mathbb{C}$ is compact.

   HINT: Consider $1/f(z)$ for $z$ near $a$.

   (iii) Suppose $f$ has a simple pole at $a$. Prove that

   $$ \lim_{r \to 0} \int_{\gamma_r} f(z)dz = i\pi \text{Res}(f,a), $$

   where $\gamma_r(t) := re^{it} + a$ for $r > 0$ and $0 \leq t \leq \pi$.

2. The holomorphic function $f$ has an isolated singularity at $z_0$ and for some $a, M \in \mathbb{R}$ satisfies

   $$ |f(z)| \leq M|z-z_0|^a $$

   near $z_0$. Show that

   (i) $z_0$ is a removable singularity if $a > -1$ and

   (ii) if $a$ satisfies $-n < a \leq -1$ for some $n \in \mathbb{N}$, then $z_0$ is a pole of order at most $n - 1$.

   HINT: Use the Cauchy estimates.

3. Suppose $\int_\gamma f = 0$ for all piecewise smooth loops $\gamma$ in a region $\Omega$ and for all holomorphic functions $f$ in $\Omega$. Show how to construct a holomorphic logarithm for any given zero-free holomorphic function in $\Omega$.

4. Explain why the identity function $f(z) := z$ has no holomorphic logarithm in $\Omega := \{z \in \mathbb{C} : 0 < |z| < 1\}$.

5. $\Omega$ is an open subset of $\mathbb{C}$ and $f: \Omega \to \mathbb{C}$ satisfies $e^{f(z)} = z$ for all $z \in \Omega$ and is continuous on $\Omega$. Show that $f$ is in fact holomorphic in $\Omega$.

6. $f$ is holomorphic by not constant in a neighborhood of $a$ and $f(a) = 0$. Show that $a$ must be an isolated zero; that is, if $r > 0$ is small enough, $f$ has no zero in $\{0 < |z-a| < r\}$.

   HINT: Look at the Taylor series of $f$ at $a$.

7. (i) The entire function $f$ satisfies

   $$ z^{-n}f(z) $$

   is bounded for some $n \in \mathbb{N}$.

   Show that $f$ is a polynomial of degree no greater than $n$.

   HINT: Cauchy estimates.

   (ii) The entire function $F$ satisfies

   $$ |F(z)| \to \infty \text{ as } |z| \to \infty. $$

   Show that $F$ is a polynomial.

   HINT: $F$ has only finitely many zeros (proof?). So for an appropriate polynomial $P, f := P/F$ satisfies the hypothesis of (i) and it has no zeros.