Instructions: You are given 10 problems from which you are to do 8. Please indicate those 8 problems which you would like to be graded by circling the problem numbers on the problem sheet. Note: All rings in this exam are associative and with 1 and all integral domains are commutative.

1. Let $G$ be a finite abelian group.
   (a) State what it means for $G$ to be an elementary abelian $p$-group, where $p$ is a prime number.
   (b) If $G$ is an elementary abelian $p$-group, explain fully in what sense can $G$ be regarded as an $\mathbb{F}_p$-vector space, where $\mathbb{F}_p$ is the field of $p$ elements.

2. Let $G = \langle x, y \rangle$ be a finite group, where $x, y$ are involutions. Prove that $G$ has a normal subgroup of index 2. (Look at $H = \langle xy \rangle$.)

3. Let $G$ be a group acting on the set $\Omega$. Assume that $\omega \in \Omega$, set $H = \text{Stab}_G(\omega)$, and assume that $K$ is a subgroup of $G$ acting transitively on $\Omega$. Prove that $G = KH$.

4. Let $R$ be a commutative ring and assume that $M, M_1, M_2, \ldots, M_r$ are maximal ideals of $R$ with $M_1M_2 \cdots M_r \subseteq M$. Prove that for some $i$, $M_i = M$.

5. Let $R = \{\frac{a}{b} \in \mathbb{Q} \mid 2 \nmid b\}$, a subring of the rational number field $\mathbb{Q}$. Show that $R$ has a unique maximal ideal, and find it.

6. Let $R$ be a ring and let $M$ be an $R$-module. Assume that $M_1, M_2 \subseteq M$ with $M = M_1 \oplus M_2$. Prove or give a counterexample to the assertion: If $N \subseteq M$ is a submodule, then
   \[ N = N \cap M_1 \oplus N \cap M_2. \]
7. Let $\alpha = \sqrt{2} + \sqrt{2} \in \mathbb{C}$. Given that $m_{\alpha, \mathbb{Q}}(x) = x^4 - 4x^2 + 2$, and that the roots of $f(x) = m_{\alpha, \mathbb{Q}}(x)$ are $\alpha = \alpha_1 = \sqrt{2} + \sqrt{2}, \alpha_2 = -\sqrt{2} + \sqrt{2}, \alpha_3 = \sqrt{2} - \sqrt{2}, \alpha_4 = -\sqrt{2} - \sqrt{2}$, answer the following:

(a) Compute the degree of the splitting field $\mathbb{K}$ over $\mathbb{Q}$ of $f(x)$.
(b) Show that the Galois group $\text{Gal}(\mathbb{K}/\mathbb{Q})$ is cyclic.

8. Let $\mathbb{F} = \mathbb{F}_q$ be the finite field of $q (= p^r)$ elements, where $p$ is prime, and let $\mathbb{K} = \mathbb{F}_{q^3} \supset \mathbb{F}$. Say that elements $\alpha, \beta \in \mathbb{K}$ are equivalent if they have the same minimal polynomial over $\mathbb{F}$. Clearly this is an equivalence relation on $\mathbb{K}$. Compute the number of equivalence classes in $\mathbb{K}$ as a function of $q$. (Hint: this is extremely easy.)

9. Let $T : V \to V$ be a linear transformation on a finite dimensional vector space over the field $\mathbb{F}$. Suppose that $T$ has the following invariant factors:

$$1 + x, \ x^2(1 + x), \ x^2(1 + x)(1 + x + x^2).$$

Answer the following questions:

(a) What is $\dim_{\mathbb{F}} V$?
(b) Is $T$ injective?
(c) What is the minimal polynomial of $T$
(d) Does $T$ have a Jordan canonical form over $\mathbb{F}$ with respect to an appropriate basis of $V$? (If this depends on the field give an example of a field $\mathbb{F}$, for which the answer is “yes,” and find the Jordan canonical form.)

10. Let $\mathbb{F}$ be a field. If $V$ is a finite-dimensional $\mathbb{F}$-vector space and if $T : V \to V$ is a linear transformation, we have the notion of minimal polynomial $m_T(x) \in \mathbb{F}[x]$ of $T$. Likewise, if $\mathbb{K} \supset \mathbb{F}$ is a finite field extension, and if $\alpha \in \mathbb{K}$, then we also have the notion of minimal polynomial $m_{\alpha}(x) \in \mathbb{F}[x]$ of the field elements $\alpha$. These notions of minimal polynomial share many similarities except that $m_{\alpha}(x)$ is always irreducible, whereas $m_T(x)$ need not be irreducible. Prove this.