Algebra Qualifying Exam
Spring 1997

All rings in this exam are associative and with 1 and all integral domains are commutative.

1. Let $G$ be a group and let $H$ be a subgroup of finite index in $G$. Show that the subgroup $N = \cap_{g \in G} gHg^{-1}$ has finite index in $G$.

2. Let $G$ be a finite group and $H$ a subgroup of $G$. Show that if $H \neq G$, then $G \neq \cup_{g \in G} gHg^{-1}$.
   Find a counter-example to this statement of infinite groups by considering a matrix group over the field of complex numbers.

3. Let $R$ be a commutative ring and let $I_1, I_2, \ldots, I_n$ be ideals of $R$. If $P$ is a prime ideal of $R$ and $\cap_{i=1}^n I_i \subseteq P$, then there is an $i$ such that $I_i \subseteq P$.

4. Let $R$ be a commutative ring. An ideal $Q \subseteq R$ is said to be a primary ideal if $ab \in Q$ and $a \notin Q$ implies that $b^n \in Q$ for some positive integer $n$. Prove that if $Q \subseteq R$ is a primary ideal, then the set $P = \{r \in R \mid r^n \in Q \text{ for some positive integer } m\}$, is the smallest prime ideal of $R$ that also contains $Q$.

5. Let $R$ be a ring and let $M$ be a left $R$-module. Then $S = \text{Hom}_R(M, M)$ is also an associative ring with 1, relative to pointwise addition and composition of homomorphisms. Show that $M$ is indecomposable if and only if $S$ has no idempotents except 0 and 1. (An element $e$ in a ring is called an idempotent if $e^2 = e$.)

6. Let $R$ be a commutative ring with 1 and $S = M_n(R)$ be the ring of all $n \times n$-matrices with entries in $R$ with matrix addition and multiplication. For any left $R$-module $M$, then $M^\oplus n = M \oplus M \oplus \cdots \oplus M$ ($n$ terms) is a left $S$-module via $A \cdot \sum_i^n m_i = \sum_i^n \sum_j^n a_{ij} m_j \in M^\oplus n$, where $A = (a_{ij})$. For each pair of indices $i, j$ we let $e_{ij} \in S$ be the matrix with a 1 in the $(i, j)$-position, and zero elsewhere.
   
   (a) Show that for any left $S$-module $N$, then, $M = e_{11}N$ is a left $R$-module.
   
   (b) Show that as $S$-modules, $N \cong M^\oplus n$.

7. Let $V$ and $W$ be two vector spaces over a field $k$. A bilinear form $f : V \times W \to k$ is called non-degenerate if for any $v \in V$ and $w \in W$, $f(v, W) = 0$ implies that $v = 0$ and $f(V, w) = 0$ implies that $w = 0$. Show that if $V$ and $W$ are finite dimensional, then a bilinear form $f$ is non-degenerate if and only if $\dim_k V = \dim_k W = n$ and there exist bases $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$ of $V$ and $W$ respectively, such that $f(v_i, w_j) = \delta_{ij}$ for all $i, j = 1, \ldots, n$.

8. Let $V$ be a vector space over a field $k$ and $T : V \to V$ be a linear transformation. Show that $f(AB)A = Af(TA)$ for any polynomial $f(x) \in k[x]$ and any linear transformation $A : V \to V$.

9. Let $K$ be a Galois extension of a field $k$ and let $F$ be a subfield of $K$ containing $k$. Show that the subgroup $H = \{g \in \text{Gal}(K/k) \mid g(F) = F\}$ is the normalizer of $\text{Gal}(K/F)$ in $\text{Gal}(K/k)$.

10. Let $K$ be the splitting field of the polynomial $x^p^2 - t \in F[x]$ over $F = \mathbb{F}_p(t)$ for a prime $p$ and an indeterminate $t$. Prove that $[K : F] = p^2$. 