Instructions: You are given 10 problems from which you are to do 8. Please indicate those 8 problems which you would like to be graded by circling the problem numbers on the problem sheet. Note: All rings in this exam are associative and with 1 and all integral domains are commutative. \( \mathbb{Z} \) and \( \mathbb{Q} \) are the sets of the integers and rational numbers respectively.

1. Let \( G \) be a finite group and \( p \) be the smallest prime divisor of \( |G| \). If \( H \) is a subgroup of \( G \) of index \( p \) in \( G \), show that \( H \) is a normal subgroup.

2. Let \( p \) and \( q \) be prime numbers. Show that any group of order \( p^2q \) is solvable.

3. Let \( D = \mathbb{Z}[i] \), the ring of Gaussian integers. Compute the order of the quotient ring \( D/(1+2i)D \).

4. Let \( f : R \rightarrow S \) be a homomorphism of rings, and let \( I \subseteq R \) be an ideal. Is it true that \( f(I) \) is an ideal of \( S \)? Prove, or give a counterexample. What if \( f \) is assumed to be surjective?

5. Let \( B \) be a ring. An ideal \( I \) of \( R \) is called nilpotent if there exists a positive integer \( n \) such that \( I^n = 0 \) (\( I^n = II \cdots I \)). Show that \( IM = \{0\} \) for any simple \( R \)-module \( M \).

6. Let \( R \) be a ring and \( M \) an (left) \( R \)-module. An element \( m \) in \( M \) is called a torsion element if \( rm = 0 \) for some \( 0 \neq r \in R \). Let \( M_t \) be the set of all torsion elements in \( M \). Show that, if \( R \) is an integral domain, then \( M_t \) is an \( R \)-submodule and the quotient module \( M/M_t \) has no torsion elements other than 0.
7. Let $V$ be a finite dimensional vector space over an algebraically closed field $F$ and $T : V \to V$ be a linear transformation. For each $a \in F$, we define $V_a = \{ v \in V \mid (T - aI)^n v = 0 \text{ for some positive integer } n \}$, which is a $T$-invariant subspace of $V$. Here $I$ is the identity linear transformation. Show the following:

(a). $V_a \neq \{0\}$ if and only if $a$ is an eigenvalue of $T$.

(b). Let $\Pi$ be the set of all eigenvalues of $T$. Then $V = \oplus_{a \in \Pi} V_a$.

8. Let $V$ be a finite dimensional vector space over a field $F$ and $A, B : V \to V$ be two commuting linear transformations. If both $A$ and $B$ are diagonalizable, then there exists a basis of $V$ such that both $A$ and $B$ have diagonal matrices with respect to this basis.

9. Let $E$ be the splitting field $f(x) = (x^3 - 2)(x^2 + x + 1)$ over $\mathbb{Q}$. Compute the Galois group $\text{Gal}(E/\mathbb{Q})$.

10. Let $E$ be the splitting field of $f(x) = x^5 - 2$ over the field $\mathbb{F}_5$, the field of 5 elements. Is $E$ a Galois extension over $\mathbb{F}_5$? Justify your answer. If your answer is “yes”, compute the Galois group.